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The viscous incompressible flow inside a cone

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The steady, axisymmetric, converging motion of a viscous incompressible fluid inside an infinite right circular cone is considered. It is shown that the exact solution of the Navier–Stokes equation for the stream function ψ is of the form $\psi(r,\theta) = AF(r\nu/A, \theta)$, where (r,θ) are spherical polar co-ordinates chosen so $r = 0$ is the apex and $\theta = 0$ is the axis of the cone, $2\pi A$ is the volumetric flow rate, and ν the kinematic viscosity of the fluid. Asymptotic expansions of the stream function are found for large and small $r\nu/A$.

For large $r\nu/A$, Stokes's method for slow motions is generalized to obtain a complete asymptotic expansion. Except for cones of special angles, all terms in this expansion may theoretically be found.

For small $r\nu/A$ a solution is constructed in two parts, namely, an inner expansion which starts from boundary-layer type equations as well as the no-slip condition at the wall, and an outer expansion in unstretched variables $r\nu/A$ and $\cos\theta$ which satisfies the boundary conditions at the axis of the cone. The condition that the inner solution merge with the outer solution with an exponentially small error requires an outer solution near the apex which is not potential sink flow, as might perhaps have been expected from the solution for two-dimensional flow in a wedge. The simplest outer flow satisfying the requirement is a vortex motion. Complete inner and outer expansions are developed and it is shown that they contain only six undetermined constants which must be determined by joining this solution numerically to the Stokes solution upstream. The inclusion of logarithmic terms in these expansions has not been found necessary.

1. Introduction

We are concerned with a theoretical investigation of viscous incompressible converging flow inside a cone. Harrison (1920) was motivated by Hamel's (1916) radial flow solution for a wedge to seek a similar solution for a cone, but found that such a solution could not satisfy the Navier–Stokes equations unless the inertial terms were neglected in the manner of Stokes. He found a solution of Stokes's equation valid for very slow motions which might be encountered far from the apex.

The difference in the two- and three-dimensional flows is a result of the way the viscosity appears in the exact solutions. Introduce a system of spherical polar co-ordinates (r,θ,ϕ), and denote the velocity components in these direc-

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tions by (u, v, w) respectively. The axis of an infinite cone with semi-vertex angle α is chosen to lie along the polar axis with the apex at the origin (figure 1). Consider a fluid with constant density ρ and viscosity μ' and steady-state solutions with axial symmetry and a zero azimuthal component of velocity, i.e. $\partial/\partial t = \partial/\partial\phi = w = 0$. In § 2 the problem is formulated mathematically, and it is shown that the exact solution for the stream function ψ is of the form

$$\psi(r,\theta) = AF(r\nu/A, \theta, \alpha), \tag{1.1}$$

where ν $(= \mu'/\rho)$ is the kinematic viscosity, and $2\pi A$ is the volumetric flow rate. The result also follows from dimensional analysis.

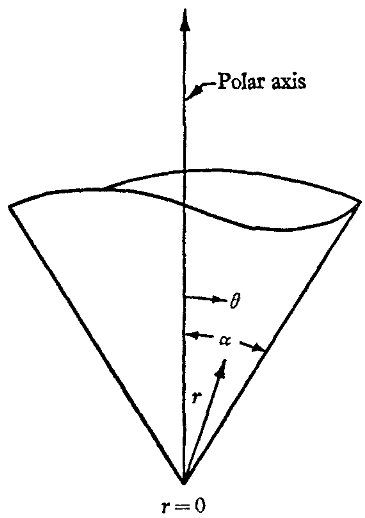

FIGURE 1. Sketch of the geometry.

The relevant variables and parameters are ψ, r, θ, μ', ρ, A, and α. The dimensionless stream function ψ/A must be a function of dimensionless combinations of the independent variables and parameters. Only one dimensionless variable involving r can be constructed; it is $r\nu/A$. Thus, ψ/A must be a function of $r\nu/A$, θ, and α, which is the result (1.1).

In two dimensions ψ/A must again depend on dimensionless combinations of variables and parameters, but A is now a volumetric flow rate per unit breadth. Thus, ν/A is dimensionless, and r can no longer be non-dimensionalized by any combinations with these parameters. Hence, ψ/A must be a function of θ, α, and ν/A, and the only solutions possible in a wedge result in purely radial flow.

To interpret (1.1) note that the exact solution depends on r only through the dimensionless variable

$$\xi = r\nu/A. \tag{1.2}$$

Hence $r \to 0$ with ν fixed is equivalent to $\nu \to 0$ with r fixed, and $r \to \infty$ with ν fixed is equivalent to $\nu \to \infty$ with r fixed. $1/\xi$, which is proportional to $1/\nu$, may be considered a local Reynolds number, and for a given viscosity we might expect to find motion characterized by a core flow and a boundary layer in the neighbourhood of the apex $(\xi \to 0)$ and creeping flow, equivalent to highly viscous motion, far from the apex ($\xi \to \infty$). In this paper asymptotic solutions of the Navier–Stokes equations are found for large and small ξ. Because the product $r\nu$ appears in the exact solution, the solutions found here are at the same time co-ordinate and parameter type expansions (Chang 1961).

Methods of approximating solutions of the Navier–Stokes equations are known for very small or very large viscosities. Conventional boundary-layer theory, which is concerned with the former, may be extended so that complete asymptotic solutions of the Navier–Stokes equations may be found for large Reynolds numbers (Kaplun 1954; Lagerstrom & Cole 1955; Goldstein 1956, 1960). In extended boundary-layer theory a solution is constructed in two parts, viz. an inner expansion which satisfies a no-slip condition at a wall, and an outer expansion which satisfies boundary conditions outside the shear layer. In addition, the inner solution must merge with the outer solution in the following way: let y be the perpendicular distance measured from the wall to a point in the fluid, and let $\eta = y/\nu^n$ ($n > 0$) be the stretched variable (or some multiple of it not involving ν) in the boundary-layer solution. The merging condition requires that the difference between the inner and outer solutions, when $\eta \to \infty$ in the former and $y \to 0$ in the latter, tend to zero. Our present knowledge of these expansions indicates that this difference must be exponentially small to have a consistent theory of approximation which can be improved step by step (for a more detailed discussion of this matter see Goldstein 1965). This will be assumed throughout this paper.

These methods of approximation raise question of existence, uniqueness, and stability, but they must be overlooked at this time to obtain useful results. Here no attempt is made to establish with rigour the asymptotic nature of the solutions obtained, yet it is shown that complete formal solutions can be found. The results of the analysis are not applicable to diverging flow where motion takes place against an adverse pressure gradient and only converging flow will be considered.

In § 3 an extended Stokes expansion is developed which represents the exact solution asymptotically for $\xi \to \infty$. The first term in this expansion is the Stokes solution found by Harrison. Four higher-order terms have been obtained, the last two by numerical integration. It is shown by induction that all terms in this expansion may be found.

A determination of the flow near the apex requires a knowledge of the core flow near the axis of the cone. The solution for a wedge (Goldstein 1938, p. 143) suggests at first sight that this flow might be that of a potential sink. However, the condition $\nu \to 0$ in the wedge solution occurs uniformly for all r, whereas in the cone the motion far from the apex is one of high viscosity no matter how small ν is, provided it is greater than zero; i.e. the fluid near the apex has come from a region where viscous effects were enormous, and there is no *a priori* reason to believe that vorticity has not diffused throughout the fluid.

In § 4 all possible boundary-layer solutions are considered for which the radial velocity at the edge of the boundary layer $U \propto r^s$. When $s = -2$ (this corresponds to potential sink flow outside the boundary layer) the radial velocity of the boundary-layer solution asymptotes its value at the edge of the boundary layer with an algebraically small error; this violates our assumption of exponential smallness. When $s \leqslant -3$ boundary-layer solutions satisfying the exponential smallness requirement can be found. Reasons are given for choosing $s = -3$. § 5 is devoted to the development of an outer expansion which has the proper behaviour at the wall and which satisfies the equations of motion and boundary conditions at the axis of the cone. It is shown that all terms in this expansion may theoretically be found, and the number of arbitrary constants introduced at each successive step is determined.

In § 6 an inner expansion is developed which matches the outer expansion with an exponentially small error. These expansions are shown to contain only six undetermined constants. Finally, § 7 is devoted to a discussion of the result.

2. Mathematical formulation of the problem and similarity properties of the exact solution

(a) Formulation

In terms of the notation introduced in § 1 the equations of continuity and momentum are (Goldstein 1938)

$$\frac{1}{r^2}\frac{\partial(r^2u)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(v\sin\theta)}{\partial\theta} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\theta}-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\nabla^2u-\frac{2u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial\theta}-\frac{2v\cot\theta}{r^2}\right], \tag{2.2}$$

and

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial\theta}+\frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+\nu\left[\nabla^2v+\frac{2}{r^2}\frac{\partial u}{\partial\theta}-\frac{v}{r^2\sin^2\theta}\right], \tag{2.3}$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right). \tag{2.4}$$

Let

$$\mu = \cos\theta \quad (0 \leqslant \theta \leqslant \pi,\ 1 \geqslant \mu \geqslant -1). \tag{2.5}$$

The continuity equation is satisfied identically by using a Stokes stream function ψ such that

$$u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = -\frac{1}{r^2}\frac{\partial\psi}{\partial\mu}, \tag{2.6}$$

and

$$v = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{r(1-\mu^2)^{\frac{1}{2}}}\frac{\partial\psi}{\partial r}. \tag{2.7}$$

For axisymmetric flow the vorticity has only an azimuthal component ζ given by

$$\zeta = \frac{1}{r}\frac{\partial(rv)}{\partial r}-\frac{1}{r}\frac{\partial u}{\partial\theta} = -\frac{D^2\psi}{r(1-\mu^2)^{\frac{1}{2}}}, \tag{2.8}$$

where

$$D^2 = \frac{\partial^2}{\partial r^2}+\frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\mu^2}. \tag{2.9}$$

Eliminating the pressure between (2.2) and (2.3) and using (2.6) and (2.7) yields

$$-(1-\mu^2)\frac{\partial[\psi,\zeta/r(1-\mu^2)^{\frac{1}{2}}]}{\partial(r,\mu)} = \nu D^4\psi, \tag{2.10}$$

or

$$\frac{1}{r^2}\frac{\partial(\psi,D^2\psi)}{\partial(r,\mu)}+\frac{2D^2\psi}{r^2}\left(\frac{\mu}{1-\mu^2}\frac{\partial\psi}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial\mu}\right) = \nu D^4\psi, \tag{2.11}$$

in the usual notation.

Boundary conditions require the fluid velocity to vanish at the wall, which may be specified by $\mu = \beta = \cos\alpha$, i.e.

$$\frac{\partial\psi}{\partial\mu} = \frac{\partial\psi}{\partial r} = 0 \quad \text{at} \quad \mu = \beta \quad \text{for} \quad r > 0. \tag{2.12}$$

ψ is constant along the wall, and it is convenient to take its value there equal to zero, or

$$\psi(r, \beta) = 0. \tag{2.13}$$

In the steady state the flux of fluid crossing every section of the cone must be the same, namely $2\pi A$. Hence

$$2\pi A = -\int_0^\alpha ur^2\,d\Omega = -2\pi\int_1^\beta r^2\frac{1}{r^2}\frac{\partial\psi}{\partial\mu}\,d\mu = 2\pi\psi(r,1). \tag{2.14}$$

[$d\Omega$ is the differential solid angle $= 2\pi\sin\theta\,d\theta = -2\pi\,d\mu$.] Therefore,

$$\psi(r, 1) = A. \tag{2.15}$$

Finally, we assume the flow far from the apex is purely radial. (2.7) then implies

$$\psi(r,\mu) \sim \psi(\mu) \quad \text{as} \quad r \to \infty \quad \text{for} \quad 1 \geqslant \mu \geqslant \beta. \tag{2.16}$$

(2.11), (2.12), (2.13), (2.15) and (2.16) constitute the mathematical formulation of this problem.

(*b*) *Similarity*

An infinite cone has no characteristic length so a conventional Reynolds number cannot be defined. Introducing the dimensionless variable ξ into (2.11), (2.12), (2.13), (2.15) and (2.16) yields:

$$\frac{1}{\xi^2}\frac{\partial(\psi, D^2\psi)}{\partial(\xi,\mu)} + \frac{2D^2\psi}{\xi^2}\left(\frac{\mu}{1-\mu^2}\frac{\partial\psi}{\partial\xi} + \frac{1}{\xi}\frac{\partial\psi}{\partial\mu}\right) = AD^4\psi, \tag{2.17}$$

$$\left(\frac{\partial\psi}{\partial\mu}\right)_{\mu=\beta} = \psi(\xi,\beta) = 0 \quad \text{for} \quad \xi > 0, \tag{2.18}$$

$$\psi(\xi, 1) = A, \tag{2.19}$$

$$\psi(\xi,\mu) \sim \psi(\mu) \quad \text{as} \quad \xi \to \infty \quad \text{for} \quad 1 \geqslant \mu \geqslant \beta. \tag{2.20}$$

D^2 is the operator of (2.9) with r replaced by ξ. If a solution exists, as is assumed, it must be of the form

$$\psi/A = F(\xi, \mu, \beta), \tag{2.21}$$

where F is a function of ξ, μ, and cone angle only.

3. Stokes flow

From the boundary condition (2.16) u and v vanish at an infinite distance from the apex. Thus a region of a Stokes flow is expected, and (1.2) and (2.21) justify the equivalence of large r with large ν.

By neglecting the non-linear terms in (2.17) a Stokes solution satisfying all the boundary conditions was found by Harrison (1920). This solution corresponds to purely radial flow, and the neglected terms in the radial velocity equation are of $O[(\mu-\beta)/\xi^5]$, so the approximation is more accurate for large ξ and regions close to the wall. Stokes's method can be generalized to obtain an asymptotic solution for large ξ in the form of a series whose lowest-order term coincides with Harrison's solution. Thus, for large ξ,

$$\psi(\xi,\mu) \sim \psi_s(\xi,\mu) = A \sum_{n=0}^{n=k} \frac{f_n(\mu)}{\xi^n}.\dagger \tag{3.1}$$

The corresponding velocity components are

$$u_s = -\frac{\nu^2}{A\xi^2}\sum_{n=0}^{n=k}\frac{f_n'(\mu)}{\xi^n} \quad \text{and} \quad v_s = \frac{\nu^2}{A\xi^2(1-\mu^2)^{\frac{1}{2}}}\sum_{n=0}^{n=k}\frac{nf_n(\mu)}{\xi^n}. \tag{3.2a,b}$$

If (3.1) is formally substituted in (2.17) and coefficients of like powers of ξ are equated to zero, a set of ordinary differential equations is obtained which determine the f_n's. The equation for the nth term is

$$(1-\mu^2)^2 f_n^{\text{iv}} - 4\mu(1-\mu^2)f_n''' + 2(n+1)(n+2)(1-\mu^2)f_n'' + (n+3)(n+2)(n+1)nf_n$$
$$= \mathscr{F}_n(\mu, f_0, f_1, \ldots, f_{n-1}), \tag{3.3}$$

where $\mathscr{F}_n$ is a function of μ and the f_n's up to f_{n-1}. The explicit form of $\mathscr{F}_n$ and the general expression for the pressure are given in Appendix A, § 1.

The boundary conditions (2.18) are satisfied if

$$f_n(\beta) = f_n'(\beta) = 0 \quad (n \geqslant 0). \tag{3.4}$$

For v_s to be finite at $\mu = 1$,

$$f_n(1) = 0 \quad (n > 0). \tag{3.5}$$

The flux condition (2.19) requires

$$f_0(1) = 1. \tag{3.6}$$

Complementary solutions of (3.3) which are finite and analytic at $\mu = 1$ are

$$W_n(\mu) = \int_1^\mu P_n(\eta)\,d\eta, \quad \text{and} \quad W_{n+2}(\mu) = \int_1^\mu P_{n+2}(\eta)\,d\eta,$$

where P_n is the Legendre polynomial of degree n. If a particular solution $y_n(\mu)$ of (3.3) is found for $n > 0$, which is analytic in the neighbourhood of $\mu = 1$, with $y_n(1) = 0$, then the solution for $f_n(\mu)$ can be written

$$f_n(\mu) = y_n(\mu) + \{[y_n'(\beta)W_{n+2}(\beta) - y_n(\beta)W_{n+2}'(\beta)]W_n(\mu)$$
$$+ [y_n(\beta)W_n'(\beta) - y_n'(\beta)W_n(\beta)]W_{n+2}(\mu)\}/J_n(\beta), \tag{3.7}$$

where
$$J_n(\beta) \equiv W_n(\beta)W_{n+2}'(\beta) - W_n'(\beta)W_{n+2}(\beta). \tag{3.8}$$

This solution satisfies the differential equation (3.3), the boundary conditions (3.4) and (3.5) and is analytic in the neighbourhood of $\mu = 1$. It may be shown by induction that solutions $f_n(\mu)$, which satisfy (3.3), (3.4) and (3.5) may always be found provided $0 < \beta < 1$. To facilitate this argument we use the following properties of $J_n(\beta)$, which are established in Appendix A, § 2.

† Generally this series and others which appear throughout this paper are asymptotic. The stream functions ψ_s, ψ_c, and ψ_{b-l} (the last two are defined in § 5 and § 6) and the physical quantities derived therefrom are defined as the sum of a finite number (say k) of terms. For numerical calculations k will be stated in each context.

For $n > 0$, $J_n(\beta)$ has a simple zero at $\beta = 0$, double zeros at $\beta = \pm 1$, and is an odd function of β. In addition, $J_n(\beta)$ vanishes nowhere in the interval $-1 \leqslant \beta \leqslant 1$ except at these points. $J_0(\beta)$ is a special case which has a double zero at $\beta = 1$, a simple zero at $\beta = -\frac{1}{2}$, and is neither even nor odd.

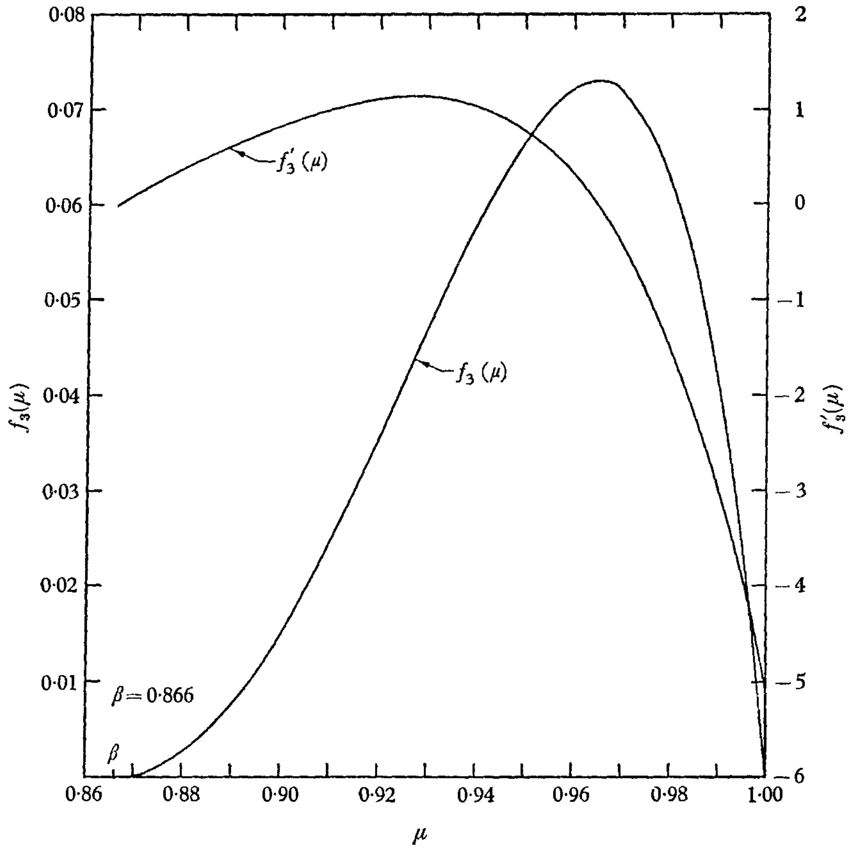

FIGURE 2. $f_3(\mu)$ and $f_3'(\mu)$ for $\beta = 0{\cdot}866$.

Solutions for f_0, f_1, and f_2 have been found analytically. They are

$$f_0(\mu) = \tfrac{1}{3}B(\mu-\beta)^2(\mu+2\beta), \tag{3.9}$$

$$f_1(\mu) = \tfrac{1}{36}B^2(1-\mu^2)(\mu-\beta)^2[2\mu-(5\beta^2-3)/\beta], \tag{3.10}$$

and

$$\begin{aligned} f_2(\mu) = \tfrac{1}{36}B^3\{&(1-\mu)(\mu-\beta)^2[b_0+b_1\mu+b_2\mu^2+b_3\mu^3+b_4\mu^4] \\ &-\mu(1-\mu^2)[b_5+b_6(7\mu^2-3)]\ln[(1+\mu)/(1+\beta)] \\ &+b_7\mu(1-\mu^2)(\mu^2-\beta^2)\}, \end{aligned} \tag{3.11}$$

where $B = 3/[(1-\beta)^2(1+2\beta)]$ and the constants $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7$, which are functions of β alone, are given in Appendix A, § 4. It can be verified by inspection that these solutions are well behaved except when $\beta = 0, \pm 1, -\frac{1}{2}$. Assume $f_{n-1}, f_{n-2}, \ldots, f_3$ are analytic for $1 \geqslant \mu \geqslant \beta\,(0 < \beta < 1)$ and they satisfy (3.3), (3.4) and (3.5); then it is easily verified that $\mathscr{F}_n(\mu)$ is analytic in this same range and $\mathscr{F}_n(\mu = 1) = 0$. Assume a series solution for $y_n(\mu)$ about $\mu = 1$ of the form

$$y_n(\mu) = a_1(1-\mu)+a_2(1-\mu)^4+\ldots.$$

This solution formally satisfies (3.3) with $y_n(1) = 0$ and $a_1 \neq 0$ (see Appendix A, § 3 for the values of a_1 and a_2). From the general theory of linear differential equations (Agnew 1942; Ince 1956) the series converges up to the nearest singular point of the differential equation, which in any case is no nearer $\mu = 1$ than $\mu = 0$. Using this $y_n(\mu)$, the solution $f_n(\mu)$ is found from (3.7) and is analytic for $1 \geqslant \mu \geqslant \beta$ $(0 < \beta < 1)$. Solutions for f_3 and f_4 have been obtained numerically by Ackerberg (1962) for $\beta = 0{\cdot}866$, and are plotted along with their first derivatives in figures 2 and 3.

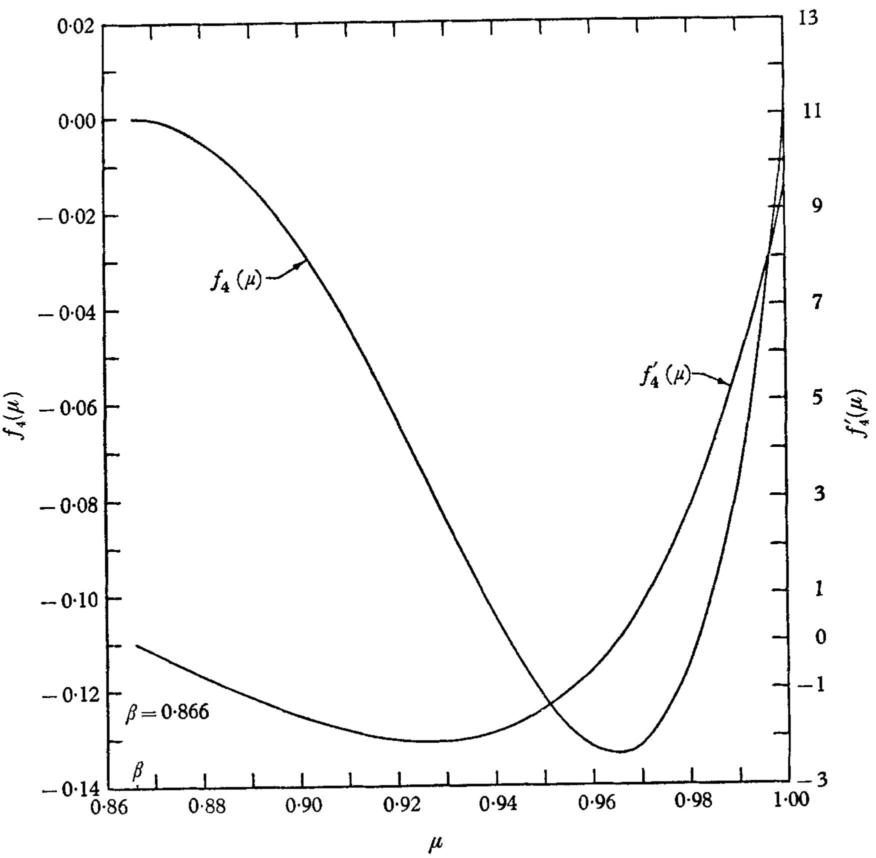

FIGURE 3. $f_4(\mu)$ and $f_4'(\mu)$ for $\beta = 0{\cdot}866$.

After f_0, the f_n's seem to alternate in sign with $f_1 > 0$. Graphs of the skin friction and radial velocity along the axis have been plotted for $\beta = 0{\cdot}866$ in figures 4 and 5. The different curves in each figure correspond to taking 3, 4, and 5 terms in the asymptotic series ψ_s, and it can be seen that at $\xi = 2{\cdot}0$, the differences are less than 6 %. It is reasonable to suppose these values are fairly accurate provided $\xi > 2{\cdot}0$.

The stream function of the Stokes solution has been plotted versus μ for different values of ξ in figures 6 and 7 for $\beta = 0{\cdot}866$. In figure 6, four terms were retained in the stream function, whereas five terms were used in figure 7. In

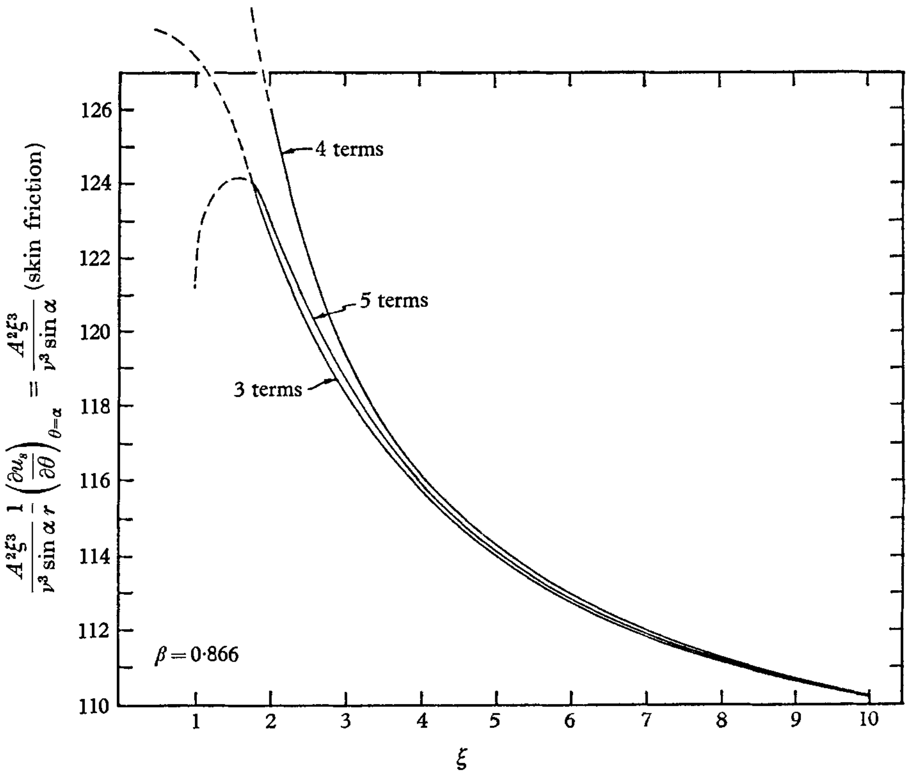

FIGURE 4. Skin friction using 3, 4, and 5 terms of $\psi_s(\mu)$ for $\beta = 0.866$.

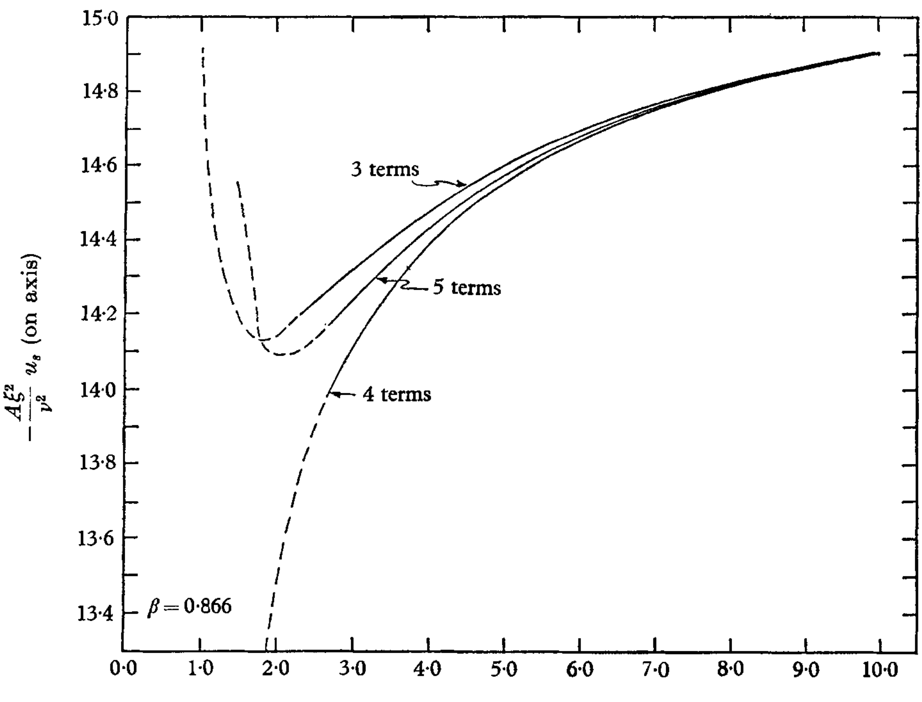

FIGURE 5. Velocity on axis using 3, 4, and 5 terms of $\psi_s(\mu)$ for $\beta = 0.866$.

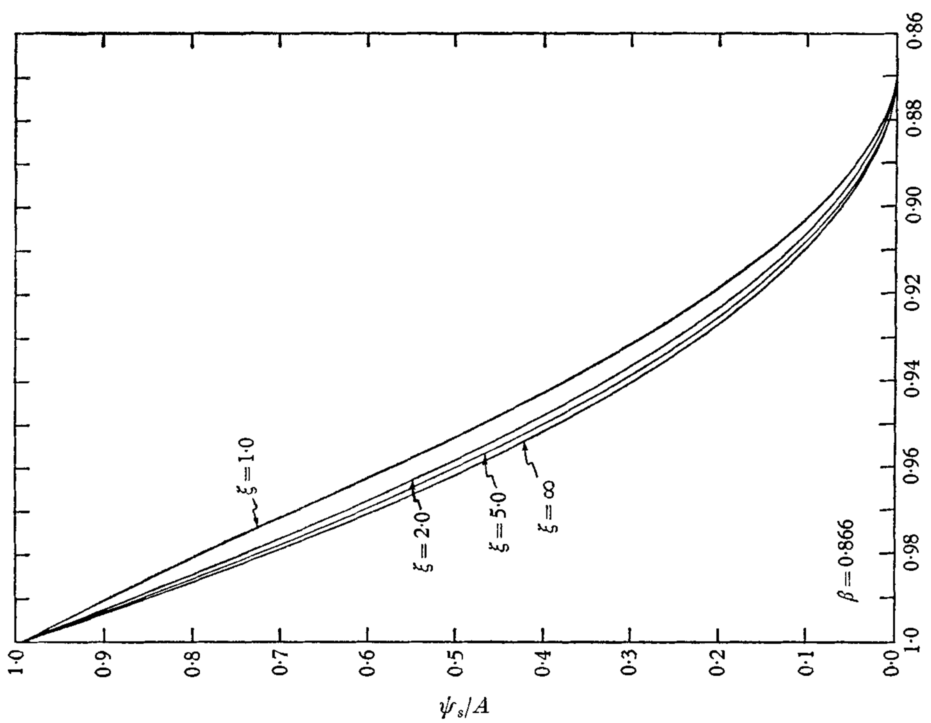

FIGURE 6. Stream function ψ_s (four terms) for $\beta = 0{\cdot}866$.

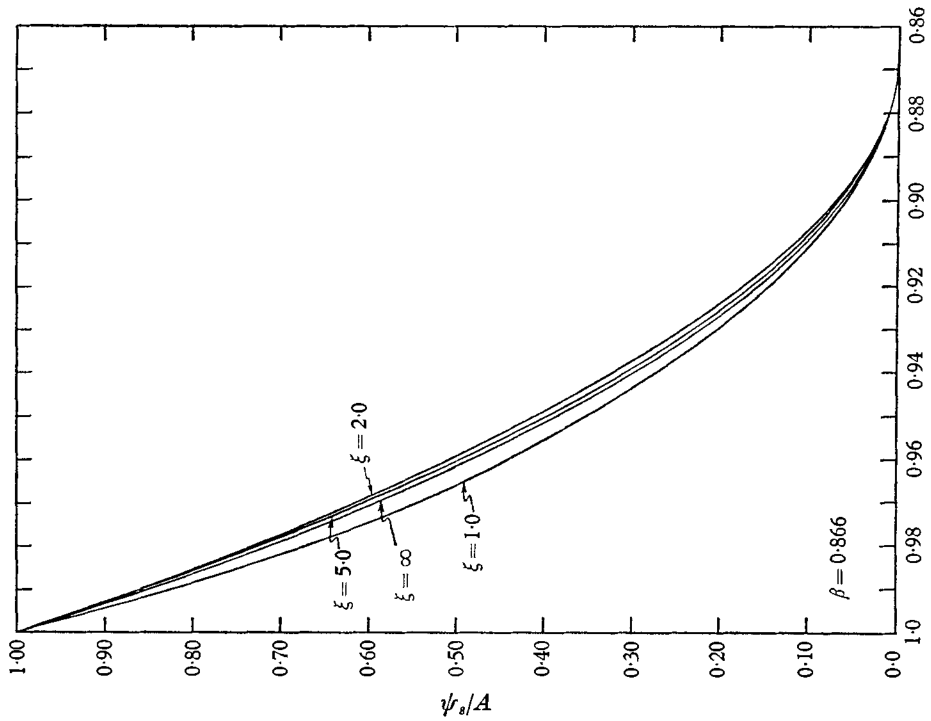

FIGURE 7. Stream function ψ_s (five terms) for $\beta = 0{\cdot}866$.

both figures, for $\xi > 2{\cdot}0$, the almost radial streamlines deviate toward the wall (see figure 8) as they are followed in from infinity.

It is interesting to note a phenomenon similar to 'Whitehead's paradox' (Proudman & Pearson 1957) for the case $\beta = 0$. From (3.9), $f_0(\mu) = \mu^3 (B = 3)$; however, solutions $f_n(\mu)$ $(n > 0)$ cannot be found which satisfy the boundary conditions because $J_n(\beta = 0) = 0$. It is possible that for $\beta \leqslant 0$ the Navier–Stokes equations might not have any solutions with purely radial flow at infinity. This conjecture is supported by experiments of Bond (1925) on wide-angled cones $(\alpha \geqslant 90°)$, which were performed to verify certain theoretical deductions from

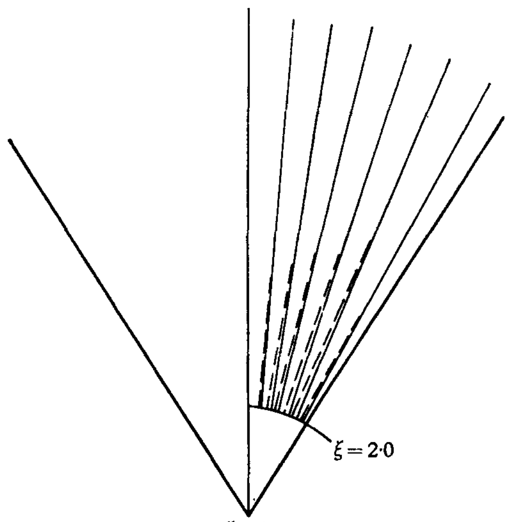

FIGURE 8. Deviation of streamlines from purely radial flow. ---, Purely radial flow; ——, curved streamlines from Stokes solution ψ_s (curvature is exaggerated).

the first term of the Stokes solution. Using f_0 only, the expressions for the radial velocity and radial pressure gradient can be obtained from (3.2*a*) and Appendix A, § 1, i.e.

$$u_s = -\frac{3\nu^2}{A(1-\beta)^2(1+2\beta)}\frac{\mu^2-\beta^2}{\xi^2}, \tag{3.12}$$

and

$$\frac{\partial p_s}{\partial r} = -\frac{6\rho\nu^5}{A^3(1-\beta)^2(1+2\beta)}\frac{1-3\mu^2}{\xi^4}. \tag{3.13}$$

Equation (3.13) predicts a reversal of the radial pressure gradient for $\theta > \theta_0 = 54° \, 45'$ when $\alpha > \theta_0$. For $\alpha > \frac{1}{2}\pi$, (3.12) predicts zero radial velocity at $\theta = \pi - \alpha$, so that for $\alpha < \frac{2}{3}\pi$ $(\beta > -\frac{1}{2})$ the velocity near the wall is radially outward with converging flow near the axis, whereas for $\alpha > \frac{2}{3}\pi$ the situation is reversed.

Using a cone with $\alpha = \frac{1}{2}\pi$, Bond made radial velocity measurements in the range $100\pi \leqslant \xi \leqslant 1000\pi$ to determine the validity of (3.12). His plot of $r^2 A u_s$ vs θ does not agree well with the theoretical curve, and photographs of the streamlines indicate marked deviations from purely radial flow. Bond also performed experiments with cones whose half angles were 110°, 141° and 160° to verify that $\partial p_s/\partial r < 0$ along the walls. Although this was verified, Bond points out from his photographs that the flow field far from the apex is certainly not radial.

4. The search for a core flow

Near the apex an outer or core flow is anticipated with a boundary layer near the wall to satisfy the no-slip condition. The core flow is not known *a priori* (unlike the wedge) and all likely solutions must be considered. For our purposes this can be accomplished by assuming the radial velocity of the core flow at the wall may be represented by a series expressed in powers (not necessarily integral) of the distance from the apex measured along the wall. A determination of the first term in the boundary-layer expansion requires the retention of the largest term in this series when $r \to 0$. Thus, if U is the radial velocity of the core flow along the wall it is sufficient to assume $U \propto r^s$ (the proportionality factor may depend on ν).

The boundary-layer equations in spherical polar co-ordinates may be found from the equations of continuity and momentum in § 2 after a new variable $\varphi = \alpha - \theta$ (which measures the angle to a variable point from the wall) is introduced in place of θ. The transformed equations are

$$\frac{\partial u}{\partial r} + \frac{2u}{r} - \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\cot(\alpha-\varphi)}{r} v = 0, \tag{4.1}$$

$$u\frac{\partial u}{\partial r} - \frac{v}{r}\frac{\partial u}{\partial \varphi} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} - \frac{\cot(\alpha-\varphi)}{r^2}\frac{\partial u}{\partial \varphi} + \frac{2}{r^2}\frac{\partial v}{\partial \varphi} - \frac{2v\cot(\alpha-\varphi)}{r^2}\right], \tag{4.2}$$

and

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \varphi} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} - \frac{\cot(\alpha-\varphi)}{r^2}\frac{\partial v}{\partial \varphi} - \frac{2}{r^2}\frac{\partial u}{\partial \varphi} - \frac{v}{r^2\sin^2(\alpha-\varphi)}\right]. \tag{4.3}$$

Define δ as the boundary-layer thickness and assume the usual orders of magnitude for boundary-layer theory, i.e.

$$\delta \ll 1, \quad \text{and} \quad \delta \to 0 \quad \text{as} \quad \nu \quad \text{or} \quad r\nu/A \to 0,$$

$$u/U = O(1), \quad r^{-1}\partial/\partial\varphi = O(1/\delta), \quad \text{and} \quad \partial/\partial r \quad \text{and} \quad \partial^2/\partial r^2 = O(1).$$

From two-dimensional boundary-layer theory we expect $v = O(U\delta/r)$. If v is larger than this the retention of the dominant term in the continuity equation (4.1) as $\nu \to 0$ yields $\partial v/\partial\varphi = 0$, and v cannot satisfy the boundary condition at the wall except in the trivial case $v \equiv 0$. (The subsequent introduction of a stream function makes further consideration of the continuity equation unnecessary.)

Using this order of magnitude, an examination of each term in (4.2) as $\nu \to 0$, shows the largest terms on each side will be of the same order of magnitude if

$$\delta = O([r\nu/U]^{\frac{1}{2}}). \tag{4.4}$$

Thus (4.2) becomes, upon retaining the largest terms,

$$u\frac{\partial u}{\partial r}-\frac{v}{r}\frac{\partial u}{\partial \varphi}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\nu}{r^2}\frac{\partial^2 u}{\partial \varphi^2}. \tag{4.5}$$

Using (4.4) in (4.3) makes all velocity terms of $O(U^2\delta)$ or less. Therefore

$$\frac{1}{\rho r}\frac{\partial p}{\partial \varphi}=O\left(\frac{\delta U^2}{r^2}\right). \tag{4.6}$$

[Thus, the pressure does not vary across the boundary layer to lowest order.]

The variation of pressure along the boundary layer is determined from the core flow solution at the wall, which is a streamline to first order for that motion. Thus, setting $v = 0$ and neglecting the viscous stress terms, (4.2) becomes

$$U\frac{\partial U}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{4.7}$$

where we have replaced u by U, the radial velocity in the core flow at the wall.

Similar solutions

Transforming the boundary-layer equation (4.5) to the independent variables (ξ, μ) yields

$$u\frac{\partial u}{\partial \xi}-\frac{(1-\mu^2)^{\frac{1}{2}}}{\xi}v\frac{\partial u}{\partial \mu}=-\frac{1}{\rho}\frac{\partial p}{\partial \xi}+\frac{\nu^2}{A\xi^2}(1-\mu^2)^{\frac{1}{2}}\frac{\partial}{\partial \mu}\left[(1-\mu^2)^{\frac{1}{2}}\frac{\partial u}{\partial \mu}\right]. \tag{4.8}$$

Similar solutions can be found by assuming a stream function of the form

$$\psi = A\xi^m h\{(\mu-\beta)/\xi^n\} = A\xi^m h(\tau), \tag{4.9}$$

where we expect $m, n \geqslant 0$. As usual in boundary-layer theory, the similarity variable was chosen so the co-ordinate normal to the wall is magnified by dividing it by a positive power of ν. The velocity components in terms of this stream function are

$$u = -(\nu^2/A^2\xi^2)\,\partial\psi/\partial\mu = -(\nu^2/A)\,\xi^{m-n-2}h'(\tau), \tag{4.10}$$

and

$$v = -\frac{\nu^2}{A^2}\frac{1}{\xi(1-\mu^2)^{\frac{1}{2}}}\frac{\partial \psi}{\partial \xi}=-\frac{\nu^2}{A}\frac{\xi^{m-2}[mh(\tau)-n\tau h'(\tau)]}{[1-(\xi\tau+\beta)^2]^{\frac{1}{2}}}. \tag{4.11}$$

Substituting these expressions in (4.8) and retaining only the largest part of the viscous stress term for $n > 0$ when $\xi \to 0$ we obtain

$$\xi^{2m-2n-1}[(m-n-2)\,h'^2 - mhh''] = -(A^2\xi^4/\rho\nu^4)\,\partial p/\partial\xi-(1-\beta^2)\,\xi^{m-3n}h'''. \tag{4.12}$$

Let the radial velocity in the core flow at the wall be given by

$$U = -(E\nu^2/A)\,\xi^s. \tag{4.13}$$

(The factor ν^2 is necessary if the solution is of the form (2.21) as shown.) Negative values of E cannot be excluded from consideration for it is possible that the radial velocity in the core flow could be negative near the axis and positive at the wall. The only indication so far that this is not the case is the deviation of the streamlines toward the wall for small ξ as predicted by the Stokes solution. Combining (4.13) with (4.7) yields

$$-\rho^{-1}\,\partial p/\partial\xi = (E^2\nu^4/A^2)\,s\xi^{2s-1}. \tag{4.14}$$

With this substitution (4.12) becomes

$$\xi^{2m-2n-1}[(m-n-2)h'^2-mhh''] = E^2s\xi^{2s+3}-(1-\beta^2)\xi^{m-3n}h'''. \tag{4.15}$$

When
$$m = \tfrac{1}{2}(s+3) \quad \text{and} \quad n = -\tfrac{1}{2}(s+1) \tag{4.16}$$

(4.15) becomes an ordinary differential equation in τ, i.e.

$$sh'^2-\tfrac{1}{2}(s+3)hh'' = E^2s-(1-\beta^2)h'''. \tag{4.17}$$

The velocity components must vanish at $\mu = \beta$; hence,

$$h(0) = h'(0) = 0. \tag{4.18}$$

In addition, the radial velocity of the boundary-layer solution must (for $\tau \to \infty$) asymptote to the radial velocity in the core flow at the wall. Using (4.10) and (4.13) and noting $m-n-2 = s$, from (4.16), this condition is

$$u \sim -(E\nu^2/A)\xi^s \quad \text{or} \quad h'(\infty) = E. \tag{4.19}$$

When $s \neq -3$, put

$$\tau^* = [|s+3|\,|E|/2(1-\beta^2)]^{\frac{1}{2}}\tau, \tag{4.20}$$

and
$$F(\tau^*) = (\operatorname{sign} E)[|s+3|/2(1-\beta^2)|E|]^{\frac{1}{2}}h(\tau). \tag{4.21}$$

(4.17) becomes

$$F''' + FF'' - \beta'(F'^2-1) = 0 \quad \text{when} \quad \operatorname{sign}[(s+3)E] < 0, \tag{4.22}$$

and
$$F''' - FF'' + \beta'(F'^2-1) = 0 \quad \text{when} \quad \operatorname{sign}[(s+3)E] > 0, \tag{4.23}$$

where $\beta' = 2s/(s+3)$. The boundary conditions (4.18) and (4.19) require

$$F(0) = F'(0) = 0, \quad \text{and} \quad F'(\infty) = 1. \tag{4.24a,b}$$

When $s = -3$, put
$$\tau^* = [3|E|/(1-\beta^2)]^{\frac{1}{2}}\tau, \tag{4.25}$$

$$F(\tau^*) = (\operatorname{sign} E)\,[3/(1-\beta^2)|E|]^{\frac{1}{2}}h(\tau), \tag{4.26}$$

and (4.17) becomes

$$F''' + F'^2 - 1 = 0 \quad \text{when} \quad \operatorname{sign} E < 0, \tag{4.27}$$

and
$$F \;\; - F'^2 + 1 = 0 \quad \text{when} \quad \operatorname{sign} E > 0, \tag{4.28}$$

with the same boundary conditions (4.24a, b).

Solutions of (4.22) with the boundary conditions (4.24a, b) have been studied by Hartree (1937) and Coppel (1960). Goldstein (1965) considered solutions of (4.23), (4.27) and (4.28) with the same boundary conditions in connexion with 'backward boundary layers'. He pointed out that in two-dimensional boundary-layer theory the asymptotic condition (4.24b) must be satisfied with an exponentially small error. Here we assume this condition must also be satisfied for axisymmetric boundary layers. Henceforth, reference to 'solutions' of (4.22), (4.23), (4.27) and (4.28) subject to the boundary conditions (4.24a, b) will imply (4.24b) is satisfied with an exponentially small error unless otherwise stated. Using Goldstein's results, these solutions can be summarized as follows:

The case $E > 0$

For $s \geqslant 0$ there are no solutions. For $-3 < s < 0$ solutions can be found which satisfy (4.24b) with an algebraically small error. For $s = -3$, two solutions are possible depending on whether $F''(0) \gtrless 0$. For all $s < -3$ real solutions exist which are unique when $F''(0) > 0$.

The case $E < 0$

For $s > -0{\cdot}2712$ real solutions exist which are unique with $F''(0) > 0$. When $s = -0{\cdot}2712$, $F''(0) = 0$. For $-3 < s < -0{\cdot}2712$ no real solutions exist, and for $s \leqslant -3$ no solutions exist.

The possibility of potential sink flow near the apex must now be excluded from consideration, for that case (which corresponds to $E > 0$, $s = -2$) is characterized by algebraic decay. In fact, the exponential smallness requirement can only be satisfied when $s \leqslant -3$ for $E > 0$, or $s \geqslant -0{\cdot}2712$ for $E < 0$. If any portion of the flux is carried into the apex by the core flow the cases with $E < 0$ can also be excluded from consideration, for a non-zero term of $O(1/r^2)$ is necessary in U, and this would dominate the assumed leading term $U \propto r^s$ ($s \geqslant -0{\cdot}2712$) near $r = 0$.

The correspondence between the exponents in the velocities at the edge of the boundary layer in two dimensions (where $u = cx^m$, cf. Goldstein 1938, p. 140) and in axisymmetric flow (4.13) is

$$s = 3m. \tag{4.29}$$

This follows from setting

$$\beta' = 2s/(s+3) = \beta = 2m/(m+1),$$

or from Mangler's transformation (Pai 1956). The case $m = -1$ in two dimensions gives the boundary-layer solution in a wedge when $\nu \to 0$. From (4.29), the axisymmetric analogue is $s = -3$, for which

$$U = -E\nu^2/A\xi^3. \tag{4.30}$$

The weakest singularity at the apex which satisfies the exponential smallness requirement is $s = -3$. In problems where it is necessary to choose a singularity to specify a solution uniquely, it is often found that the weakest one gives the best description of the physical facts. Therefore, we assume (4.30) is the correct lowest-order term in the radial velocity of the core flow at the wall. Note $U = O(1/\nu)$ and $\delta = O(\nu)$ which is not usually the case in boundary-layer theory.

This singularity seems to violate the condition of constant flux crossing every section of the cone. In the next section it will be shown that this term gives rise to a vortex motion with closed streamlines which does not contribute to the flux. The flux condition must be imposed on higher-order terms.

5. The core flow near the apex

A core flow expansion with the correct radial velocity at the wall (to first order) is

$$\psi(\xi,\mu) \sim \psi_c(\xi,\mu) = A \sum_{n=0}^{n=k} \xi^{n-1} g_n(\mu). \tag{5.1}$$

The corresponding velocity components are

$$u_c(\xi,\mu) = -\frac{\nu^2}{A\xi^3} \sum_{n=0}^{n=k} \xi^n g_n'(\mu), \tag{5.2a}$$

and

$$v_c(\xi,\mu) = -\frac{\nu^2}{A\xi^3}\frac{1}{(1-\mu^2)^{\frac{1}{2}}} \sum_{n=0}^{n=k} (n-1)\,\xi^n g_n(\mu). \tag{5.2b}$$

For v_c to be finite on the axis it is necessary that

$$g_n(1) = 0 \quad \text{for all } n \text{ except } n = 1. \tag{5.3}$$

The value of $g_1(1)$ is found by noting that the core flow must satisfy the boundary condition at the cone axis. Equation (2.15) and the relations (5.3) require

$$g_1(1) = 1. \tag{5.4}$$

If (5.1) is substituted in (2.17) and coefficients of like powers of ξ are equated to zero, a set of third-order ordinary differential equations is obtained for the g_n's. Each equation has an immediate first integral, which is for $n \geqslant 0$

$$(1-\mu^2)\,g_n'' + (n-1)(n-2)\,g_n - 5c(1-\mu^2)\,g_0^4 g_n$$
$$= -\frac{1-\mu^2}{g_0^{n-5}(\mu)}\int_1^{\mu} \mathscr{G}_n(g_0, g_1, \ldots, g_{n-1}, \mu)\,d\mu + \frac{c_n(1-\mu^2)}{g_0^{n-5}(\mu)}. \tag{5.5}$$

The general form of $\mathscr{G}_n$ and the pressure are given in Appendix B. The constants c and c_n $(n > 0)$ are arbitrary with $c_0 = -4c$, and can be related to the vorticity distribution in the core flow.

5.1. $g_0(\mu)$

The equation for $g_0(\mu)$ is

$$(1-\mu^2)\,g_0'' + 2g_0 = c(1-\mu^2)\,g_0^5. \tag{5.6}$$

In accordance with the general theory of a core and boundary layer, the boundary is a streamline for the leading term in the core solution. Hence, $g_0(\beta) = 0$. With

$$g_0(1) = g_0(\beta) = 0, \tag{5.7}$$

$g_0(\mu)$ does not contribute to the flux of fluid through the cone, and very near the apex a motion with closed streamlines is expected.

Solutions of (5.6) with the boundary conditions (5.7) may be studied by considering the cases $c \gtreqless 0$. When $c > 0$, the series solution about $\mu = 1$ is composed only of positive terms for $\mu \leqslant 1$. Thus, once started from zero, the solution increases monotonically and cannot satisfy the boundary condition $g_0(\beta) = 0$. When $c = 0$, the only solution with $g_0(1) = 0$ is $g_0(\mu) = C(1-\mu^2)$, and this also cannot satisfy $g_0(\beta) = 0$. When $c < 0$, g_0'' and g_0 are of opposite signs for $|\mu| < 1$, and the solution oscillates in this range.

Define $G_0(\mu) = (-c)^{\frac{1}{4}} g_0(\mu)$. The equation and boundary conditions for $G_0(\mu)$ are

$$(1-\mu^2)\,G_0'' + 2G_0 = -(1-\mu^2)\,G_0^5, \tag{5.8a}$$

and

$$G_0(1) = G_0(\beta) = 0. \tag{5.8b}$$

It is sufficient to consider $G_0(\mu) \geqslant 0$ in the neighbourhood of $\mu \leqslant 1$ because (5.8a, b) are invariant when G is replaced by $-G$. In this case $G_0'(1) < 0$, and there is a value of $G_0'(1)$ for which the first zero of $G_0(\mu)$ (the zero nearest $\mu = 1$) is at $\mu = \beta$, so that $G_0(\mu) > 0$ for $\beta < \mu < 1$. When $G_0'(1)$ is less than this value, $G_0(\mu)$ oscillates more rapidly and many values of $G_0'(1)$ can be found for which $G_0(\beta) = 0$ [see figure 9]. In a sense (5.8a, b) define an eigenvalue problem where the eigenvalues correspond to the initial slopes $G_0'(1)$. We postulate that the simplest of these solutions occurs in nature, i.e. the one with the largest $G_0'(1)$. The radial velocity from this solution has one inflexion point [see figure 10 noting that to first order $u_c = -(\nu^2/A\xi^3)(-c)^{-\frac{1}{4}} G_0'(\mu)$], whereas the more complicated solutions have one less inflexion point than the number of times they are zero in the interval $\beta \leqslant \mu \leqslant 1$. It is likely that the simplest solution is the most stable.

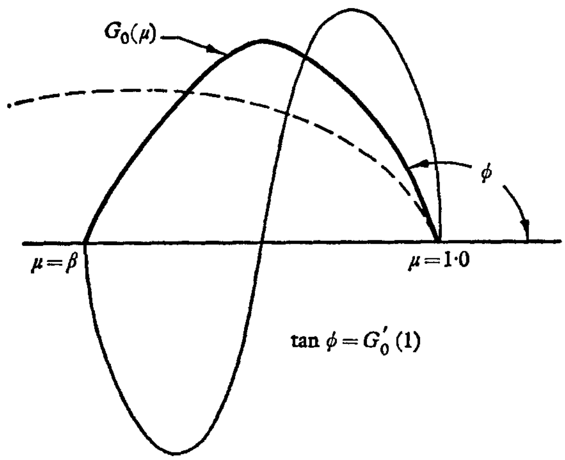

FIGURE 9. Variation of $G_0(\mu)$ with $G_0'(1)$.

A series solution for $g_0(\mu)$ about $\mu = \beta$ can be written

$$g_0(\mu) = \sum_{m=0}^{m=\infty} a_{0m}(\mu-\beta)^m, \tag{5.9}$$

where $a_{00} = 0$. In terms of $G_0(\mu)$ and a_{01}, the solution for $g_0(\mu)$ is

$$g_0(\mu) = a_{01}G_0(\mu)/G_0'(\beta), \tag{5.10}$$

and since $G_0(\mu)$ and $G_0'(\beta)$ have the same sign, the sign of $G_0(\mu)$ is irrelevant. The constant c in (5.5) is related to a_{01} by

$$c = -[G_0'(\beta)/a_{01}]^4.$$

Numerically computed values of $G_0(\mu)$ and $G_0'(\mu)$ have been obtained by Ackerberg (1962) and are plotted in figure 10 for $\beta = 0{\cdot}866$.

From (5.2a), the radial velocity at the wall to first order is

$$u_c(\xi,\beta) = -a_{01}\nu^2/A\xi^3. \tag{5.11}$$

It is necessary that $a_{01} > 0$ or the boundary-layer equation has no solution. This condition specifies the direction in which the fluid particles in the closed streamline region of the flow are moving, and this movement is in accordance with the deviation of the streamlines toward the wall predicted by the Stokes solution for small ξ (cf. figures 8 and 11).

The streamlines due to $Ag_0(\mu)/\xi$ are closed and it might be expected that Batchelor's theorem (1956) is applicable so that the vorticity is proportional to the distance from the axis of symmetry. The theorem cannot be applied in this case, however, because when $\nu \to 0$ all streamlines of finite value extend from $r = 0$ to $r \to \infty$ and by choosing an intermediate value of r and allowing $\nu \to 0$ every streamline passes infinitesimally close to the axis and the wall, which is a singular surface. Thus, the shortest distance from any streamline to a singular surface tends to zero with ν.

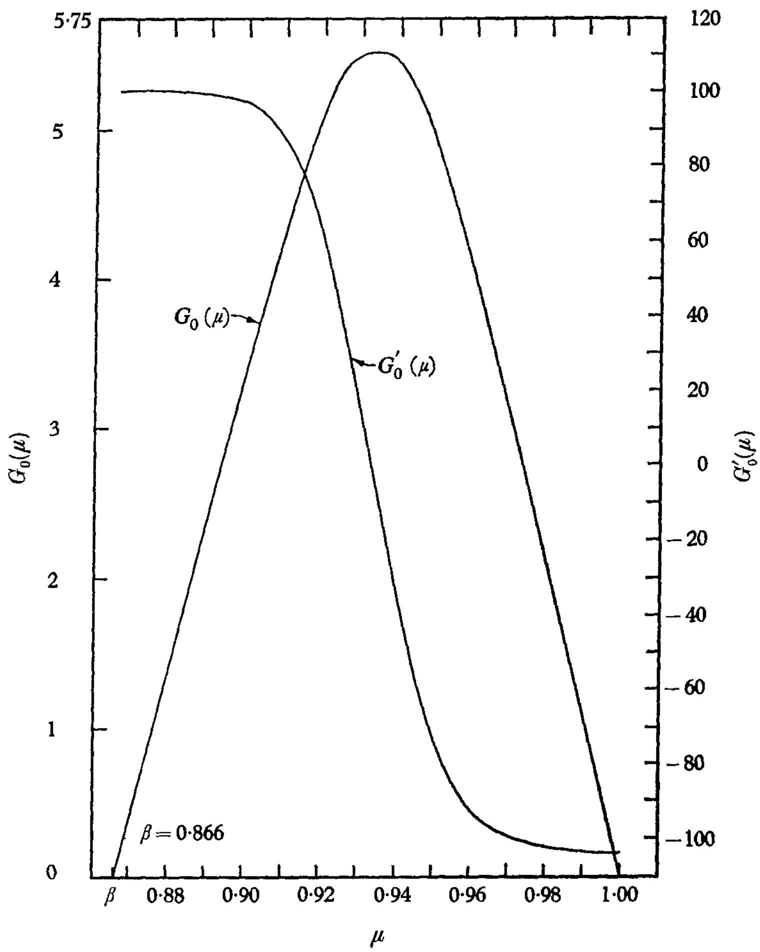

FIGURE 10. $G_0(\mu)$ and $G_0'(\mu)$ for $\beta = 0{\cdot}866$.

5.2. $g_1(\mu)$

The equation for $g_1(\mu)$ is

$$g_1'' - 5cg_0^4 g_1 = c_1 g_0^4. \tag{5.12}$$

By inspection, a particular solution is

$$g_1^{(p)} = -c_1/5c. \tag{5.13}$$

Since $-cg_0^4(\mu) = G_0^4(\mu)$, the homogeneous equation for g_1 can be written

$$g_1'' + 5G_0^4 g_1 = 0. \tag{5.14}$$

Two independent complementary solutions of this equation will be denoted by $G_{11}(\mu)$ and $G_{12}(\mu)$. They satisfy the boundary conditions:

$$G_{11}(1) = 0; \quad G'_{11}(1) = 1, \tag{5.15}$$

and

$$G_{12}(1) = 1; \quad G'_{12}(1) = 0. \tag{5.16}$$

The Wronskian of these solutions is

$$G'_{11}(\mu)\,G_{12}(\mu) - G'_{12}(\mu)\,G_{11}(\mu) = 1. \tag{5.17}$$

In the neighbourhood of $\mu = \beta$, the solution for $g_1(\mu)$ can be written

$$g_1(\mu) = \sum_{n=0}^{n=\infty} a_{1m}(\mu-\beta)^m. \tag{5.18}$$

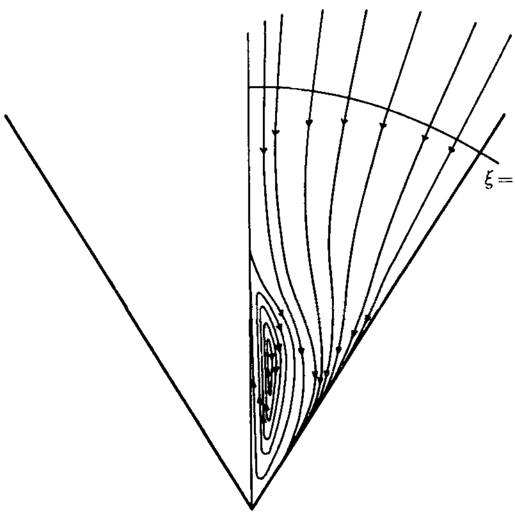

FIGURE 11. Vortex motion near apex.

In terms of $G_{11}(\mu)$, $G_{12}(\mu)$, a_{10}, and a_{11}, the solution of (5.12) with the right-hand side included and $g_1(1) = 1$ is

$$g_1(\mu) = [\{1-a_{10}+a_{11}G_{11}(\beta)\}/\{1-G'_{11}(\beta)\}]\,[1+G'_{12}(\beta)\,G_{11}(\mu)-G_{12}(\mu)]$$
$$+\,[a_{10}+a_{11}G_{12}(\beta)\,G_{11}(\mu)+(1-a_{10})\,G_{12}(\mu)], \tag{5.19}$$

for $G'_{11}(\beta) \neq 1$. When $G'_{11}(\beta) = 1$, a solution for $g_1(\mu)$ can be found only when $a_{10} = 1+a_{11}G_{11}(\beta)$; it is

$$g_1(\mu) = \tilde{c}+[a_{11}-(1-\tilde{c})\,G'_{12}(\beta)]\,G_{11}(\mu)+(1-\tilde{c})\,G_{12}(\mu), \tag{5.20}$$

where

$$\tilde{c} = \{6a_{16}+[1+a_{11}G_{11}(\beta)]\,[G'_0(\beta)]^4\}/[G'_0(\beta)]^4.$$

It should be noted that in either case two arbitrary constants appear in the solution: a_{10} and a_{11} when $G'_{11}(\beta) \neq 1$, and a_{11} and a_{16} when $G'_{11}(\beta) = 1$. Numerically computed values of $G_{11}(\mu)$, $G'_{11}(\mu)$, $G_{12}(\mu)$, and $G'_{12}(\mu)$ have been found by Ackerberg (1962), and are plotted in figures 12 and 13.

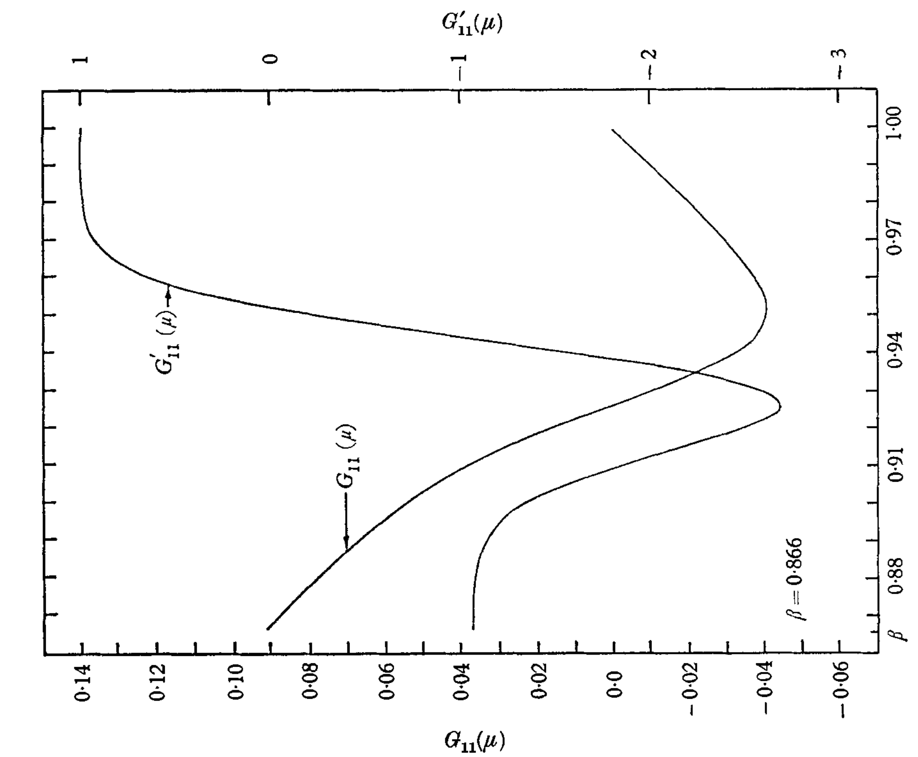

FIGURE 12. $G_{11}(\mu)$ and $G'_{11}(\mu)$ for $\beta = 0.866$.

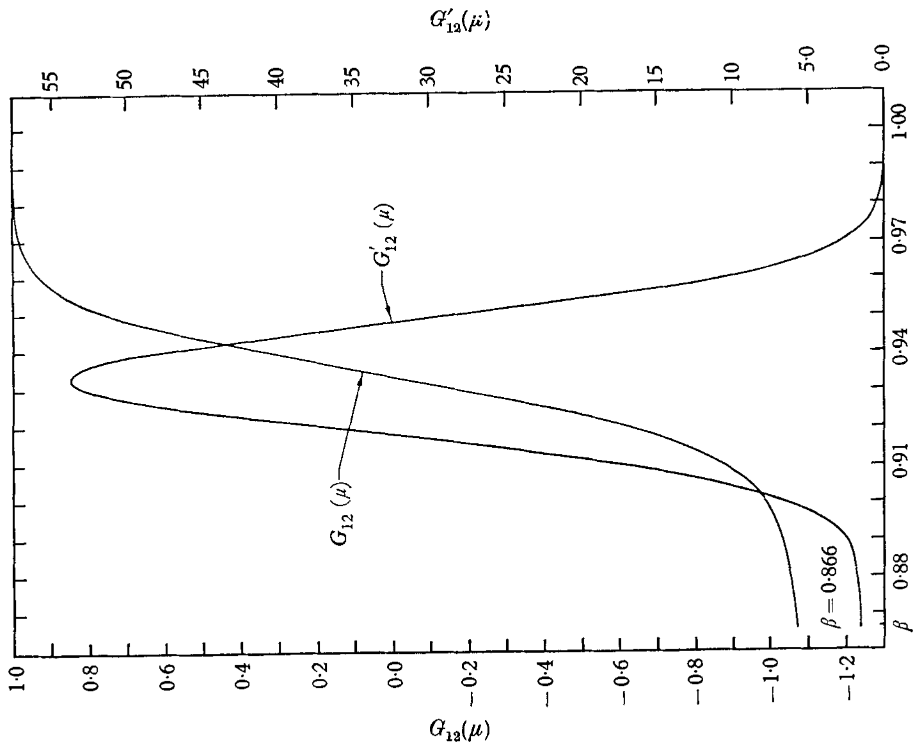

FIGURE 13. $G_{12}(\mu)$ and $G'_{12}(\mu)$ for $\beta = 0.866$.

For $G'_{11}(\beta) \neq 1$, the constant c_1 in (5.12) is

$$c_1 = 5[G'_0(\beta)/a_{01}]^4\,[1+a_{11}\,G_{11}(\beta) - a_{10}\,G'_{11}(\beta)]/[1-G'_{11}(\beta)], \tag{5.21}$$

and when $G'_{11}(\beta) = 1$,

$$c_1 = (5/a_{01}^4)\,\{6a_{16} + [1+a_{11}\,G_{11}(\beta)]\,[G'_0(\beta)]^4\}. \tag{5.22}$$

5.3. $g_2(\mu)$

The equation for $g_2(\mu)$ is

$$g''_2 - 5cg_0^4 g_2 = -cg_0^3 \int_1^\mu \left\{\frac{12g_0^5 + [(1-\mu^2)g_0^5]''}{g_0^4}\right\} d\mu + 2g_0^3[2c_1 g_1 + 5cg_1^2] + c_2 g_0^3. \tag{5.23}$$

The right-hand side of this equation can be written in terms of the solutions $G_0(\mu)$, $G_{11}(\mu)$, and $G_{12}(\mu)$, and particular integrals can be found which vanish for $\mu = 1$. The complementary solutions of (5.23) are the same as those for $g_1(\mu)$. However, only an arbitrary multiple of $G_{11}(\mu)$ can be added to any particular integral which vanishes for $\mu = 1$, because (5.3) requires $g_2(1) = 0$.

Therefore, barring exceptional circumstances, the solution for $g_2(\mu)$ involves two arbitrary constants which have been introduced as c_2 and the arbitrary multiple of $G_{11}(\mu)$. In the neighbourhood of $\mu = \beta$, the solution for $g_2(\mu)$ can be written

$$g_2(\mu) = \sum_{m=0}^{m=\infty} a_{2m}(\mu-\beta)^m, \tag{5.24}$$

and the two arbitrary constants can be related to a_{20}, a_{21} and the constants appearing in previous g_n's.

5.4. *Complementary solutions of* (5.5)

The solutions of (5.5) will now be investigated for $n > 2$. In general g_n can be written as a series solution about $\mu = \beta$,

$$g_n(\mu) = \sum_{m=0}^{m=\infty} a_{nm}(\mu-\beta)^m. \tag{5.25}$$

The homogeneous equation (5.5),

$$(1-\mu^2)g''_n + (n-1)(n-2)g_n - 5c(1-\mu^2)g_0^4 g_n = 0, \tag{5.26}$$

has for $n > 0$ a series solution about $\mu = 1$ of the form

$$y_1^{(n)}(\mu) = \sum_{s=0}^{s=\infty} A_s(1-\mu)^{s+1}. \tag{5.27}$$

When $n = 1$ or 2, $\mu = 1$ is not a singular point of the differential equation, and a second independent solution is

$$y_2^{(1,2)}(\mu) = \sum_{s=0}^{s=\infty} A'_s(1-\mu)^s. \tag{5.28}$$

When $n \neq 1$ or 2, $\mu = 1$ is a singular point and the second independent solution is

$$y_2^{(n)}(\mu) = By_1^{(n)}(\mu)\ln(1-\mu) + \sum_{s=0}^{s=\infty} D_s(1-\mu)^s, \tag{5.29}$$

where the constants D_s are related to the arbitrary constant B.

5.5. *Particular solutions*

From the expression for v_c in (5.2*b*), $g_n(\mu)/(1-\mu)^{\frac{1}{2}}$ must be finite at $\mu = 1$ for $n \neq 1$. If $g_n(\mu)$ is analytic near $\mu = 1$, then as $\mu \to 1$,

$$g_n(\mu) = O(1-\mu) \quad \text{for} \quad n \neq 1, \tag{5.30}$$

and, from (B 3) in Appendix B,

$$\mathscr{G}_n(\mu) = \begin{cases} O(1) & \text{for} \quad 2 \leqslant n \leqslant 5, & (5.31\,a) \\ O[(1-\mu)^{n-6}] & \text{for} \quad n \geqslant 6. & (5.31\,b) \end{cases}$$

Therefore,

$$\frac{1-\mu^2}{g_0^{n-5}} \int_1^\mu \mathscr{G}_n(\mu)\, d\mu = \begin{cases} O[(1-\mu)^{7-n}] & \text{for} \quad 2 \leqslant n \leqslant 5, & (5.32\,a) \\ O(1-\mu) & \text{for} \quad n \geqslant 6. & (5.32\,b) \end{cases}$$

A particular solution corresponding to this integral term [with c_n taken as zero in (5.5)] can be found for all n which is zero and analytic in the neighbourhood of $\mu = 1$.

The second term on the right-hand side of (5.5) has the following behaviour near $\mu = 1$:

$$c_n(1-\mu^2)/g_0^{n-5}(\mu) = O[(1-\mu)^{-n+6}]. \tag{5.33}$$

Particular solutions arising from this term which are zero and analytic in the neighbourhood of $\mu = 1$ can be found only when $n < 6$. For $n \geqslant 6$, the c_n's must be zero to satisfy the boundary condition $g_n(1) = 0$. (The vorticity is not bounded on the axis unless the c_n's are chosen in this way.)

The results for the g_n's can be summarized as follows: For $n = 0$, only one arbitrary constant a_{01} is introduced. For $n = 1$, the solution involves two arbitrary constants, a_{10} and a_{11}. For $2 \leqslant n \leqslant 5$ the solutions $g_n(\mu)$ which are zero and analytic in the neighbourhood of $\mu = 1$ introduce two arbitrary constants for each n. These constants are c_n and the arbitrary multiplier of the complementary solution $y_1^{(n)}(\mu)$. For $n \geqslant 6$, only one arbitrary constant is introduced for each n, since c_n must be zero. This single constant is the arbitrary multiplier of the complementary solution $y_1^{(n)}(\mu)$. In all cases (barring exceptional circumstances) the arbitrary constants can be related to some combination of a_{n0} and a_{n1} for $2 \leqslant n \leqslant 5$, and a_{n0} for $n \geqslant 6$. In the next section it will be shown that only six of these constants are independent.

6. The boundary-layer solution

The results of § 4 [see (4.9) and (4.16) with $s = -3$] suggest a boundary-layer expansion near the apex of the form

$$\psi(\xi, \mu) \sim \psi_{\text{b.l}}(\xi, \tau) = A \sum_{n=0}^{n=k} \xi^n h_n(\tau), \tag{6.1}$$

where $\tau = (\mu - \beta)/\xi$. The corresponding velocity components are

$$u_{\text{b.l}}(\xi, \tau) = -\frac{\nu^2}{A} \frac{1}{\xi^3} \sum_{n=0}^{n=k} \xi^n h_n'(\tau), \tag{6.2}$$

and

$$v_{\text{b.l}} = \frac{\nu^2}{A} \frac{1}{\xi^2(\omega^2 - 2\beta\xi\tau - \xi^2\tau^2)^{\frac{1}{2}}} \sum_{n=0}^{n=k} \xi^n[\tau h_n' - n h_n], \tag{6.3}$$

where $\omega^2 = 1 - \beta^2$.

Substituting (6.1) in (2.17) after it has been transformed to independent variables (ξ, τ), and equating coefficients of like powers of ξ to zero yields a set of ordinary differential equations for the h_n's. The equation for h_0 is

$$\omega^2 h_0^{\text{iv}} - 6h_0' h_0'' = 0. \tag{6.4}$$

[This is (4.17) differentiated once with $s = -3$.] The equation for h_n $(n > 0)$ is

$$\omega^2 h_n^{\text{iv}} + (n-6) h_0' h_n'' - 6h_0'' h_n' - n h_0''' h_n = \mathscr{H}_n(\tau, h_0, h_1, \ldots, h_{n-1}), \tag{6.5}$$

where $\mathscr{H}_n$ depends on τ and the h_n's up to h_{n-1}. The general form of $\mathscr{H}_n$ is given in Appendix C, § 1.

The no-slip conditions at the wall are satisfied if

$$h_n'(0) = 0 \quad \text{for} \quad n \geqslant 0, \tag{6.6a}$$

and

$$h_n(0) = 0 \quad \text{for} \quad n > 0. \tag{6.6b}$$

When $n = 0$, (2.18) requires

$$h_0(0) = 0. \tag{6.7}$$

In addition, the boundary-layer solution (as $\tau \to \infty$) must merge with the core solution (as $\mu \to \beta$) with an exponentially small error. Mathematically this requires that the boundary-layer expansion asymptote (when $\tau \to \infty$) to the outer solution rearranged in powers of ξ and τ. Using (5.1) and (5.25), the core expansion can be written

$$\psi_c(\xi, \mu) = A \sum_{n=0}^{n=\infty} \sum_{m=0}^{m=\infty} \xi^{n-1} a_{nm} (\mu - \beta)^m. \tag{6.8}$$

Substituting $\mu - \beta = \xi\tau$, and using the fact that $a_{00} = g_0(\beta) = 0$, (6.8) becomes

$$\psi_c(\xi, \tau) = A \sum_{n=0}^{n=\infty} \sum_{m=0}^{m=\infty} a_{nm} \xi^{m+n-1} \tau^m = A \sum_{n=0}^{n=\infty} \xi^n \sum_{m=0}^{m=n+1} a_{n+1-m,m} \tau^m. \tag{6.9}$$

From (6.1), the merging condition is satisfied if

$$\psi_{\text{b-l}}(\xi, \tau) = A \sum_{n=0} \xi^n h_n(\tau) \sim A \sum_{n=0} \xi^n \sum_{m=0}^{m=n+1} a_{n+1-m,m} \tau^m. \tag{6.10}$$

Equating coefficients of similar powers of ξ we finally obtain

$$h_n(\tau) \sim \sum_{m=0}^{m=n+1} a_{n+1-m,m} \tau^m, \tag{6.11}$$

where the error in (6.11) must be exponentially small. With $n = 0$,

$$h_0(\tau) \sim a_{01}\tau + a_{10}, \tag{6.12}$$

and to the first order, the radial velocity at the edge of the boundary layer is $u_{\text{b-l}}(\xi, \infty) = -(\nu^2/A\xi^3) h_0'(\infty) = -\nu^2 a_{01}/A\xi^3$, as it should be, and a_{01} must be positive.

The integration of (6.4) yields two solutions, whose difference depends on whether $h_0''(0) \gtrless 0$. When $h_0''(0) > 0$, the radial velocity in the boundary layer decreases monotonically from zero at the wall to its value at the edge of the boundary layer (see figure 14 (*a*)). This solution is

$$h_0^{(1)}(\tau) = a_{01}\tau - (6a_{01})^{\frac{1}{2}}\,\omega \tanh\left[(\tfrac{3}{2}a_{01})^{\frac{1}{2}}\tau/\omega + \tanh^{-1}\sqrt{\tfrac{2}{3}}\right] + 2\omega(a_{01})^{\frac{1}{2}}.\dagger \qquad (6.13)$$

For $h_0''(0) < 0$, the radial velocity near the boundary is positive. As τ increases, this velocity falls to zero and reverses direction so as to become converging (see figure 14 (*b*)). This solution is

$$h_0^{(2)}(\tau) = a_{01}\tau + (6a_{01})^{\frac{1}{2}}\,\omega \tanh\left[-(\tfrac{3}{2}a_{01})^{\frac{1}{2}}\tau/\omega + \tanh^{-1}\sqrt{\tfrac{2}{3}}\right] - 2\omega(a_{01})^{\frac{1}{2}}. \qquad (6.14)$$

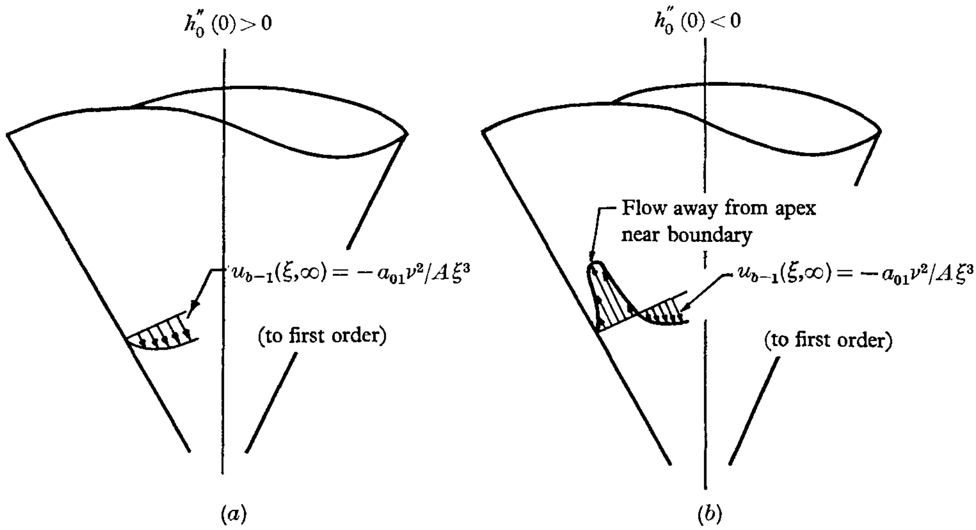

FIGURE 14. (*a*) Radial velocity profile of lowest-order boundary-layer solution when $h_0''(0) > 0$. (*b*) Radial velocity profile of lowest-order boundary-layer solution when $h_0''(0) < 0$.

Both solutions satisfy (6.12) with an exponentially small error. However, when $h_0''(0) < 0$, the skin friction near the apex is negative (a questionable result physically) and it is not clear how (6.14) could join with the Stokes solution near thew all for $\alpha < 90°$.‡ For these reasons, (6.13) will be taken as the correct boundary-layer solution and hereafter h_0 will refer to $h_0^{(1)}$. Starting with the boundary-layer equations an equivalent solution is possible for the converging flow in a wedge and is also rejected.

From (6.13), the asymptotic form of $h_0(\tau)$ is

$$h_0(\tau) \sim a_{01}\tau + \omega(a_{01})^{\frac{1}{2}}(2-\sqrt{6}), \quad \text{when} \quad \tau \to \infty. \qquad (6.15)$$

Comparison of (6.15) with (6.12) determines a_{10} in terms of a_{01}, i.e.

$$a_{10} = \omega(a_{01})^{\frac{1}{2}}(2-\sqrt{6}). \qquad (6.16)$$

a_{10} can be related to the displacement thickness of the lowest-order boundary-layer solution.

† Positive square roots are to be taken throughout this section.

‡ See the discussion following (3.13).

6.1. *Complementary solutions of* (6.5)

The coefficients of (6.5) are analytic for $0 \leqslant \tau \leqslant \infty$; thus, a series solution about $\tau = 0$ yields the four linearly independent complementary solutions

$$y_1 = 1 + b_{10}\tau^4 + b_{11}\tau^5 + \ldots + b_{1m}\tau^{m+4} + \ldots, \tag{6.17a}$$

$$y_2 = \tau + b_{20}\tau^4 + b_{21}\tau^5 + \ldots + b_{2m}\tau^{m+4} + \ldots, \tag{6.17b}$$

$$y_3 = \tau^2 + b_{30}\tau^4 + b_{31}\tau^5 + \ldots + b_{3m}\tau^{m+4} + \ldots, \tag{6.17c}$$

$$y_4 = \tau^3 + b_{40}\tau^4 + b_{41}\tau^5 + \ldots + b_{4m}\tau^{m+4} + \ldots. \tag{6.17d}$$

The asymptotic behaviour of these solutions for $\tau \to \infty$ can be found from (6.5) using asymptotic values for the coefficients, i.e.

$$\omega^2 y_m^{\text{iv}} + (n-6)\, a_{01} y_m'' = 0, \tag{6.18}$$

where we have used (6.15) and $m = 1, 2, 3, 4$. This equation has solutions

$$y_m = c_{m1} + c_{m2}\tau + c_{m3}\exp[\{(6-n)\,a_{01}\}^{\frac{1}{2}}\tau/\omega] + c_{m4}\exp[-\{(6-n)\,a_{01}\}^{\frac{1}{2}}\tau/\omega] \quad \text{for} \quad n < 6, \tag{6.19a}$$

$$y_m = d_{m1} + d_{m2}\tau + d_{m3}\tau^2 + d_{m4}\tau^3 \quad \text{for} \quad n = 6, \tag{6.19b}$$

$$y_m = e_{m1} + e_{m2}\tau + e_{m3}\cos[\{(n-6)\,a_{01}\}^{\frac{1}{2}}\tau/\omega] + e_{m4}\sin[\{(n-6)\,a_{01}\}^{\frac{1}{2}}\tau/\omega] \quad \text{for} \quad n > 6, \tag{6.19c}$$

where c_{mi}, d_{mi}, and e_{mi} are constants which depend on n.

If $c_{m3} \neq 0$ for $m = 3, 4$ when $n < 6$, the linear combination

$$Y_n(\tau) = c_{43}y_3(\tau) - c_{33}y_4(\tau) \quad (n < 6), \tag{6.20a}$$

satisfies

$$Y_n(0) = Y_n'(0) = 0, \tag{6.20b}$$

and as $\tau \to \infty$

$$Y_n(\tau) \sim C_1 + C_2\tau + O[\exp - \{(6-n)\,a_{01}\}^{\frac{1}{2}}\tau/\omega], \tag{6.20c}$$

where $C_1 = c_{43}c_{31} - c_{33}c_{41}$ and $C_2 = c_{43}c_{32} - c_{33}c_{42}$. Any multiple of Y_n will satisfy (6.20*b*) and have an asymptotic expansion similar to (6.20*c*) with different constants, C_1' and C_2'. To define uniquely such a multiple in terms of Y_n, a value for C_1' or C_2' (say C_2') may be chosen.†

6.2. *General solutions*

If we assume $h_1, h_2, \ldots, h_{n-1}$ satisfy (6.11) it is not difficult to show that $\mathscr{H}_n$ asymptotes (as $\tau \to \infty$) to a polynomial of degree $\leqslant n-1$ for $n \neq 6$, and a polynomial of degree 3 when $n = 6$ (Appendix C, § 2). These results will be used to show h_n satisfies (6.11), thereby establishing (6.11) for all n by induction. In Appendix C, § 2, particular integrals of (6.5) are found for large τ which asymptote a polynomial of degree $\leqslant n+1$; however, in general these solutions will not satisfy (6.6*a*, *b*). To find solutions which satisfy both conditions determine any particular integral, Λ_n, of (6.5) with

$$\Lambda_n(0) = \Lambda_n'(0) = 0. \tag{6.21a}$$

† If $c_{33} = c_{43} = 0$ in (6.19*a*) Y_n can be constructed by taking any linear combination of y_3 and y_4. In this case it is necessary to choose values for C_1 and C_2 to define Y_n uniquely.

From the above discussion Λ_n must asymptote a polynomial of degree $\leqslant n+1$ plus any terms arising from the complementary solutions y_m [see (6.19a, b, c)]. Thus, for $n < 6$

$$\Lambda_n(\tau) \sim \mathscr{P}_n(\tau)+\mathscr{D}_1 \exp[\{(6-n)a_{01}\}^{\frac{1}{2}}\tau/\omega]+\mathscr{D}_2 \exp[-\{(6-n)a_{01}\}^{\frac{1}{2}}\tau/\omega], \tag{6.21b}$$

where $\mathscr{P}_n(\tau)$ is a polynomial of degree $\leqslant n+1$, and $\mathscr{D}_1$ and $\mathscr{D}_2$ are constants. Another particular integral with no growing exponential term and which satisfies (6.21a) is

$$Z_n(\tau) = \Lambda_n(\tau)-(\mathscr{D}_1/c_{m3})\,y_m(\tau) \quad \text{for} \quad m = 3 \text{ or } 4 \quad (n < 6), \tag{6.22a}$$

and for large τ, Z_n asymptotes to $\overline{\mathscr{P}}_n(\tau)$ where

$$\overline{\mathscr{P}}_n(\tau) = \mathscr{P}_n(\tau)-\mathscr{D}_1(c_{m1}+c_{m2}\tau)/c_{m3} \quad \text{for} \quad m = 3 \text{ or } 4. \tag{6.22b}$$

For $n < 6$, the general solution of (6.5) which satisfies (6.6a, b) and (6.11) with an exponentially small error is

$$h_n(\tau) = Z_n(\tau)+KY_n(\tau) \quad (n < 6), \tag{6.23}$$

where K is an arbitrary constant. As $\tau \to \infty$

$$h_n(\tau) \sim \overline{\mathscr{P}}_n(\tau)+K(C_1+C_2\tau)+O[\exp-\{(6-n)a_{01}\}^{\frac{1}{2}}\tau/\omega].\dagger \tag{6.24}$$

When a few terms of (6.11) are written out,

$$h_n(\tau) \sim a_{n+1,0}+a_{n1}\tau+a_{n2}\tau^2+\ldots+a_{0,n+1}\tau^{n+1}, \tag{6.25}$$

and it is evident that a value for any one of the constants $a_{n+1,0}$, a_{n1}, or K must be chosen to uniquely specify h_n. We choose a_{n1} as the arbitrary constant and $a_{n+1,0}$ and K will be expressible in terms of a_{n1} and the arbitrary constants introduced for previous n.‡

When $n = 6$, the y_m's have no exponentially large terms when $\tau \to \infty$ (6.19b). In this case the particular integrals Λ_6 asymptote to a polynomial of degree $\leqslant 7$, and the general solution for h_6 is

$$h_6(\tau) = \Lambda_6(\tau)+K_1y_3(\tau)+K_2y_4(\tau) \quad \text{for} \quad n = 6, \tag{6.26a}$$

where K_1 and K_2 are arbitrary constants. Thus

$$h_6(\tau) \sim \mathscr{P}_6(\tau)+K_1[d_{31}+d_{32}\tau+d_{33}\tau^2+d_{34}\tau^3]+K_2[d_{41}+d_{42}\tau+d_{43}\tau^2+d_{44}\tau^3]. \tag{6.26b}$$

From (6.11) we have, for $n = 6$,

$$h_6(\tau) \sim a_{70}+a_{61}\tau+a_{52}\tau^2+a_{43}\tau^3+\ldots+a_{07}\tau^7, \tag{6.27}$$

and it can be seen that values for any two of the constants a_{70}, a_{61}, a_{52}, a_{43}, K_1, K_2 must be chosen to specify h_6 uniquely. We choose a_{52} and a_{43} as the arbitrary constants and the others are expressible in terms of these and the arbitrary constants introduced for previous n.

† C_1 and C_2 are known from (6.20c) so that K is the only unknown constant in (6.24).

‡ If $c_{33} = c_{43} = 0$ in (6.19a), $a_{n+1,0}$ and a_{n1} must both be specified to determine h_n uniquely.

Hence, for $n \leqslant 6$, the arbitrary constants introduced in the boundary-layer solutions are $a_{01}, a_{11}, a_{21}, a_{31}, a_{41}, a_{51}, a_{43}, a_{52}$. From § 5 we recall that for $2 \leqslant n \leqslant 5$, the solutions for $g_n(\mu)$ introduced two arbitrary constants for each n, which were chosen as a_{n0} and a_{n1}. Therefore, a_{43} can be expressed in terms of a_{40} and a_{41} from g_4, and since a_{40} can be expressed in terms of a_{n1} ($n < 4$) using h_3, a_{43} can be expressed in terms of a_{n1} ($n \leqslant 4$). Similarly, a_{52} can be expressed in terms of a_{51} and a_{50} from g_5, and a_{50} can be expressed in terms of a_{n1} ($n < 5$) from h_4, so that a_{52} can be expressed in terms of a_{n1} ($n \leqslant 5$). Thus, for $n \leqslant 6$, six arbitrary constants are introduced; they are a_{01}, a_{11}, a_{21}, a_{31}, a_{41}, a_{51}.

When $n > 6$, the solutions Λ_n asymptote to a polynomial of degree $\leqslant n+1$ plus oscillatory terms. Another particular integral Z_n which satisfies (6.21 a) and has no oscillatory terms when $\tau \to \infty$ can be found by adding fixed multiples of y_3 and y_4 to Λ_n. A linear combination of y_3 and y_4 [equivalent to Y_n, see (6.20a, b, c)] can no longer be found which asymptotes to a linear function plus exponentially small terms. Thus,

$$h_n(\tau) = Z_n(\tau) \quad \text{for} \quad n > 6, \tag{6.28}$$

and no arbitrary constants are introduced in these solutions except those introduced for $n \leqslant 6$. Therefore, the boundary-layer and core-flow expansions contain only six unknown constants, and all other a_{nm}'s ($n, m \geqslant 0$) can be expressed in terms of these.

These methods of construction will be used for finding h_1.

6.3. h_1

The equation for h_1 is

$$\omega^2 h_1^{\text{iv}} - 5h_0' h_1'' - 6h_0'' h_1' - h_0''' h_1 = 2\beta\tau h_0^{\text{iv}} + 4\beta h_0'''. \tag{6.29}$$

It is convenient to express h_1 in terms of universal functions. Let

$$\tau' = (a_{01}^{\frac{1}{2}}/\omega)\,\tau. \tag{6.30}$$

Then from (6.13)

$$h_0(\tau) = (a_{01})^{\frac{1}{2}}\omega H_0(\tau'), \tag{6.31}$$

where

$$H_0(\tau') = \tau' - \surd 6 \tanh\left[\surd\tfrac{3}{2}\tau' + \tanh^{-1}\surd\tfrac{2}{3}\right] + 2. \tag{6.32}$$

Define

$$2\beta H_1(\tau') = h_1(\tau). \tag{6.33}$$

Substituting these expressions in (6.29) and dropping the primes, an equation for H_1 is obtained:

$$H_1^{\text{iv}} - 5H_0' H_1'' - 6H_0'' H_1' - H_0''' H_1 = \tau H_0^{\text{iv}} + 2H_0'''. \tag{6.34}$$

The homogeneous boundary conditions $h_1(0) = h_1'(0) = 0$ apply to H_1 as well. The condition for $\tau \to \infty$ is found from (6.11) by setting $n = 1$:

$$h_1(\tau) \sim a_{20} + a_{11}\tau + a_{02}\tau^2. \tag{6.35}$$

By definition $a_{02} = \frac{1}{2}g_0''(\beta)$. From (5.6) and (5.7) $g_0''(\beta) = 0$. Therefore, using (6.30) and (6.33)

$$H_1(\tau) \sim a_{20}/2\beta + \omega a_{11}\tau/2\beta(a_{01})^{\frac{1}{2}}. \tag{6.36}$$

Note that it is unnecessary to use the properties of $g_0(\mu)$ to deduce $a_{02} \equiv 0$. This follows from the asymptotic behaviour of any particular integral of (6.34). Other checks of this form occur for $n > 1$.

The solution for h_1 can be expressed in terms of solutions to three problems. They are

1. $$y_3^{iv} - 5H_0' y_3'' - 6H_0'' y_3' - H_0''' y_3 = 0, \quad (6.37a)$$
$$y_3(0) = y_3'(0) = y_3'''(0) = 0, \quad (6.37b)$$
$$y_3''(0) = 1. \quad (6.37c)$$

2. $$y_4^{iv} - 5H_0' y_4'' - 6H_0'' y_4' - H_0''' y_4 = 0, \quad (6.38a)$$
$$y_4(0) = y_4'(0) = y_4''(0) = 0, \quad (6.38b)$$
$$y_4'''(0) = 1. \quad (6.38c)$$

These solutions are simple multiples of y_3 and y_4 discussed previously.

3. $$\Lambda_1^{iv} - 5H_0' \Lambda_1'' - 6H_0'' \Lambda_1' - H_0''' \Lambda_1 = \tau H_0^{iv} + 2H_0''', \quad (6.39a)$$
$$\Lambda_1(0) = \Lambda_1'(0) = 0, \quad (6.39b)$$
$$\Lambda_1'''(0) = 0; \quad \Lambda_1''(0) = 1; \quad \Lambda_1^{iv}(0) = 2H_0'''(0) = -6. \quad (6.39c)$$

Any other particular integral satisfying (6.39*b*) would suffice in place of Λ_1. The asymptotic representations of y_3, y_4, and Λ_1 (as $\tau \to \infty$) will all contain multiples of
$$1, \tau, e^{\sqrt{5}\tau}, e^{-\sqrt{5}\tau}.$$
Y_1 is found by taking a linear combination of y_3 and y_4 which has no growing exponential term, i.e.
$$Y_1(\tau) = y_3(\tau) + By_4(\tau), \quad (6.40a)$$
and
$$Y_1(\tau) \sim \epsilon_1 + \epsilon_2 \tau + \epsilon_4 e^{-\sqrt{5}\tau}. \quad (6.40b)$$
A multiple of y_3 is added to Λ_1 so the combination has no growing exponential term (y_4 could be used instead of y_3):
$$Z_1(\tau) = \Lambda_1(\tau) + Dy_3(\tau), \quad (6.41a)$$
$$Z_1(\tau) \sim \delta_1 + \delta_2 \tau + \delta_4 e^{-\sqrt{5}\tau}. \quad (6.41b)$$
H_1 is found by adding an arbitrary multiple of Y_1 to Z_1, or
$$H_1(\tau) = \gamma_1 Y_1(\tau) + Z_1(\tau), \quad (6.42a)$$
and as $\tau \to \infty$
$$H_1(\tau) \sim \gamma_1\epsilon_1 + \delta_1 + (\gamma_1\epsilon_2 + \delta_2)\tau + O(e^{-\sqrt{5}\tau}). \quad (6.42b)$$
This must coincide with (6.36); hence
$$\gamma_1\epsilon_1 + \delta_1 = a_{20}/2\beta,$$
and
$$\gamma_1\epsilon_2 + \delta_2 = \omega a_{11}/2\beta\sqrt{a_{01}}.$$
Therefore
$$\gamma_1 = \frac{\omega a_{11}/2\beta a_{01}^{\frac{1}{2}} - \delta_2}{\epsilon_2}, \quad (6.43)$$
and
$$a_{20} = \frac{2\beta\epsilon_1}{\epsilon_2}\left[\frac{\omega a_{11}}{2\beta a_{01}^{\frac{1}{2}}} - \delta_2\right] + 2\beta\delta_1. \quad (6.44)$$
(6.43) relates the arbitrary multiplier of $Y_1(\tau)$ to a_{11} and a_{01}, and (6.44) expresses a_{20} in terms of a_{n1} ($n < 2$). The constants ϵ_1, ϵ_2, δ_1, δ_2 are related to the known solutions Y_1 and Z_1 in the following ways:
$$\epsilon_1 = \lim_{\zeta\to\infty} [Y_1(\zeta) - Y_1'(\infty)\zeta], \quad \epsilon_2 = Y_1'(\infty)$$
and
$$\delta_1 = \lim_{\zeta\to\infty} [Z_1(\zeta) - Z_1'(\infty)\zeta], \quad \delta_2 = Z_1'(\infty).$$
Finally, the solution for $h_1(\tau)$ is
$$h_1(\tau) = [(\omega a_{11} a_{01}^{-\frac{1}{2}} - 2\beta\delta_2)/\epsilon_2]\, Y_1(a_{01}^{\frac{1}{2}}\tau/\omega) + 2\beta Z_1(a_{01}^{\frac{1}{2}}\tau/\omega). \quad (6\ 45)$$

6.4. *Summary*

The results of this section can be summarized as follows. The boundary-layer and core-flow expansions which are valid for $\xi \to 0$ involve six undetermined constants which are taken to be $a_{01}, a_{11}, a_{21}, a_{31}, a_{41}, a_{51}$. All other a_{nm}'s can be expressed in terms of these six from relationships among the a_{nm}'s through the g_n's and h_n's.

7. Discussion

The determination of the flow near the apex requires values for the six arbitrary constants in the core and boundary-layer expansions. Our analysis does not restrict these values except for the requirement $a_{01} > 0$. By assumption, the expansions found here (for small and large ξ) are valid asymptotically as $\xi \to 0$ and $\xi \to \infty$, and it is natural to require these solutions to agree numerically in any region where their validities overlap, if such regions exist. In § 3 a reference to figures 4 and 5 indicated that five terms of the Stokes solution could be used to give accurate results for $\xi > 2{\cdot}0$. Classical boundary-layer theory, however, is valid for Reynolds numbers of the order of 10^2, i.e. $\xi = O(10^{-2})$. Because of the large difference in these values and the striking change in flow character, it is unlikely that any accurate determination of the unknown constants could be obtained by numerically joining one or two terms of each expansion near $\xi = 2{\cdot}0$. In fact, such an attempt by Ackerberg (1962) failed.

Certain questions of theoretical interest warrant further investigation. In particular, are the boundary conditions at infinity well posed for cones whose half angles equal or exceed 90°, and do the boundary-layer solutions for $s < -3$ represent possible solutions in the neighbourhood of the apex?

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REFERENCES

Ackerberg, R. C. 1962 Ph.D. Thesis, Harvard University.
Agnew, R. W. 1942 *Differential Equations*. New York: McGraw-Hill.
Batchelor, G. K. 1956 *J. Fluid Mech.* **1**, 177.
Bond, W. N. 1925 *Phil. Mag.* **6**, 1058.
Chang, I-Dee 1961 *J. Math. Mech.* **10**, 811.
Coppel, W. A. 1960 *Phil. Trans.* A, **253**, 101.
Goldstein, S. 1938 *Modern Developments in Fluid Dynamics*. Oxford University Press.
Goldstein, S. 1956 *Univ. Calif. (Berkeley) Inst. Engng Res. Tech. Rep.* no. HE-150-144.
Goldstein, S. 1960 *Lectures on Fluid Mechanics*. New York: Interscience.
Goldstein, S. 1965 *J. Fluid Mech.* **21**, 33.

HAMEL, G. 1916 *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **25**, 34.
HARRISON, W. T. 1920 *Proc. Camb. Phil. Soc.* **19**, 307.
HARTREE, D. R. 1937 *Proc. Camb. Phil. Soc.* **33**, 223.
INCE, E. L. 1956 *Ordinary Differential Equations*. New York: Dover Publications.
KAPLUN, S. 1954 *Z. angew. Math. Phys.* **5**, 111.
LAGERSTROM, P. A. & COLE, J. D. 1955 *J. Rat. Mech. Anal.* **4**, 817.
PAI, SHIH I. 1956 *Viscous Flow Theory*. I. Princeton: Van Nostrand.
PROUDMAN, I. & PEARSON, J. R. A. 1957 *J. Fluid Mech.* **2**, 237.
WHITTAKER, E. T. & WATSON, G. N. 1961 *Modern Analysis* (4th edn). Cambridge University Press.

Appendix A

A.1. $\mathscr{F}_n(\mu)$

$$\mathscr{F}_0(\mu) = 0, \tag{A1}$$

$$\mathscr{F}_n(\mu) = \sum_{s=0}^{s=n-1} f'_{n-1-s}(s+4)\,[s(s+1)f_s + (1-\mu^2)f''_s]$$
$$- \sum_{s=0}^{s=n-1} (n-1-s)f_{n-1-s}\left[s(s+1)f'_s + (1-\mu^2)f'''_s + s(s+1)\frac{2\mu}{1-\mu^2}f_s\right] \quad (n \geqslant 1). \tag{A2}$$

The pressure p_s calculated from the Stokes solution stream function $\psi_s(\xi, \mu)$ is

$$\frac{A^2}{\rho\nu^4}(p_\infty - p_s) = -\frac{[(1-\mu^2)f''_0]'}{3\xi^3} + \sum_{n=0}^{n=k}\frac{1}{(n+4)\,\xi^{n+4}}\Bigg\{-(1-\mu^2)f''''_{n+1} + 2\mu f''_{n+1}$$
$$-(n+1)(n+2)f'_{n+1} - \sum_{s=0}^{s=n}\left[sf''_{n-s}f_s - (n-s+2)f'_s f'_{n-s} - s(n-s)\frac{f_{n-s}f_s}{1-\mu^2}\right]\Bigg\}, \tag{A3}$$

where p_∞ is the pressure at infinity. Unspecified arguments are at a general point μ.

A.2. *Properties of* $J_n(\beta)$

From definitions in § 3,

$$J_n = W_n W'_{n+2} - W'_n W_{n+2}, \tag{A4}$$

and

$$W'_n = P_n. \tag{A5}$$

W_n satisfies the differential equation

$$(1-\mu^2)\,W''_n + n(n+1)\,W_n = 0. \tag{A6}$$

Combining (A 5) and (A 6):

$$W_n = -\frac{(1-\mu^2)\,P'_n}{n(n+1)} \quad (n \neq 0). \tag{A7}$$

(A 4) can be written for $n \neq 0$:

$$J_n = -(1-\mu^2)\left[\frac{P'_n P_{n+2}}{n(n+1)} - \frac{P'_{n+2}P_n}{(n+2)(n+3)}\right]. \tag{A8}$$

The following identities are known for Legendre Polynomials (Whittaker & Watson 1961):

$$(n+1)\,P_{n+1} - (2n+1)\,\mu P_n + nP_{n-1} = 0, \tag{a}$$
$$nP_n - \mu P'_n + P'_{n-1} = 0, \tag{b}$$
$$(n+1)\,P_n - P'_{n+1} + \mu P'_n = 0, \tag{c}$$

and

$$P'_{n+1} - P'_{n-1} - (2n+1)\,P_n = 0. \tag{d}$$

It is desirable to express P_{n+2} and P'_{n+2} in terms of P_n and P'_n. Replace n by $n+1$ in (b) and substitute for P'_{n+1} from (c) to obtain

$$P_{n+1} = \mu P_n - \{(1-\mu^2)/(n+1)\} P'_n. \tag{e}$$

In (d) replace n by $n+1$ and substitute for P_{n+1} from (e):

$$P'_{n+2} = P'_n[1-(2n+3)(1-\mu^2)/(n+1)] + (2n+3)\mu P_n. \tag{A 9}$$

In (a) replace n by $n+1$ and substitute for P_{n+1} from (e):

$$(n+2) P_{n+2} = P_n[(2n+3)\mu^2 - (n+1)] - \{(2n+3)/(n+1)\}\mu(1-\mu^2) P'_n. \tag{A 10}$$

Substitute (A 9) and (A 10) into (A 8) for P_{n+2} and P'_{n+2} to obtain for $n \neq 0$

$$J_n = \frac{(2n+3)(1-\mu^2)}{(n+3)(n+2)(n+1)n}\left[n(n+1)\mu P_n^2 + \frac{n+3}{n+1}\mu(1-\mu^2) P_n'^2 + (1-3\mu^2) P_n P'_n\right]. \tag{A 11}$$

By similar manipulations it is possible to eliminate P_n and P'_n from (A 8) and express J_n in terms of P_{n+2} and P'_{n+2}. For $n \neq 0$ this result is

$$J_n = \frac{(2n+3)(1-\mu^2)}{(n+3)(n+2)(n+1)n} \times \left[(n+3)(n+2)\mu P_{n+2}^2 + \frac{n}{n+2}\mu(1-\mu^2) P_{n+2}'^2 + (1-3\mu^2) P_{n+2} P'_{n+2}\right]. \tag{A 12}$$

Differentiate (A 4) to obtain

$$J'_n = W_n W''_{n+2} - W''_n W_{n+2}. \tag{A 13}$$

Noting that $W''_n = P'_n$, and using (A 6), (A 13) can be written

$$J'_n = -\left\{\frac{2(2n+3)}{(n+3)(n+2)(n+1)n}\right\}(1-\mu^2) P'_n P'_{n+2}. \tag{A 14}$$

The Legendre polynomials P_n are even or odd depending on their index n being even or odd. The products P_n^2, $P_n'^2$ are even, whereas products of the form $P_n P'_n$ are odd. From (A 11) or (A 12) it is evident that J_n is an odd function of μ and hence vanishes at $\mu = 0$. [In fact, when n is even, J_n has a factor μ^3, see (A 14).] The factors $(1-\mu^2)$ in (A 11), or (A 12), and (A 14) indicate that J_n has double zeros for $\mu = \pm 1$.

Differentiate (A 14) and set $\mu = 1$ to obtain

$$J''_n(1) = \left\{\frac{4(2n+3)}{(n+3)(n+2)(n+1)n}\right\} P'_n(1) P'_{n+2}(1) = 2n+3. \tag{A 15}$$

Therefore, $J_n > 0$ in the neighbourhood of $\mu = 1$. We will now show that J_n $(n > 0)$ vanishes only at the points $\mu = 0, \pm 1$.

Assume J_n has a zero in the interval $I\,(0 < \mu < 1)$. Then since it is a continuous function which is zero at the endpoints, its derivative must pass through zero at least twice in I. From (A 14) the zeros of J'_n coincide with those of P'_n and P'_{n+2}. J_n is initially positive in the neighbourhood of $\mu = 1$, becomes negative by hypothesis, and returns to zero at $\mu = 0$. Therefore, for at least one zero of J'_n, J_n must be negative (it cannot be zero as will be shown). The values of J_n when J'_n vanishes are given by (A 11) and (A 12) with $P'_n = P'_{n+2} = 0$, i.e.

$$J_n = \left\{\frac{(2n+3)}{(n+3)(n+2)}\right\} \mu(1-\mu^2) P_n^2 \quad \text{when} \quad P'_n = 0, \tag{A 16}$$

and

$$J_n = \left\{\frac{(2n+3)}{(n+1)n}\right\} \mu(1-\mu^2) P_{n+2}^2 \quad \text{when} \quad P'_{n+2} = 0. \tag{A 17}$$

Thus, $J_n > 0$ at all points where $J'_n = 0$ in I. Hence, the assumption that J_n has a zero in I must be incorrect. It is impossible for $J_n = J'_n = 0$ at a point in I because P_n and P'_n never vanish simultaneously. Since J_n is odd this result is also true for $-1 < \mu < 0$.

J_0 is given by

$$J_0 = (\mu-1)^2(\mu+\tfrac{1}{2}), \tag{A 18}$$

and this function has a double zero for $\mu = 1$ and a simple zero at $\mu = -\frac{1}{2}$.

A.3. *Values for a_1 and a_2 in the particular solution $y_n(\mu)$ for $n > 0$*

$$a_1 = \frac{1}{(n+3)(n+2)(n+1)n} \sum_{s=0}^{n-1} (s+4) f'_{n-1-s}[2f''_s - s(s+1) f'_s], \tag{A 19}$$

$$a_2 = \frac{1}{288}\left\{\sum_{s=0}^{n-1}(s+4)\left[s(s+1)f'_s f''_{n-1-s} + \frac{(s+2)(s-1)}{2} f''_s f'_{n-1-s} - 2f''_{n-1-s} f''_s - 2f'_{n-1-s} f'''_s\right] + \sum_{s=0}^{n-1}(n-1-s)\left[2f'_{n-1-s} f'''_s - \frac{s(s+1)}{2} f'_{n-1-s} f''_s + \frac{s(s+1)}{2} f'_{n-1-s} f'_s\right]\right\}. \tag{A 20}$$

All arguments are at $\mu = 1$.

A.4. *The constants in* (3.11)

$$b_0 = \left(\frac{16}{7}\beta^4 - \frac{5}{4}\beta^3 - \frac{96}{35}\beta^2 + \frac{7}{4}\beta + \frac{112}{165} - \frac{1}{2\beta}\right)\Big/\beta^2, \tag{A 21}$$

$$b_1 = \left(\frac{3587}{2772}\beta^5 - \frac{359}{99}\beta^4 - \frac{91559}{13860}\beta^3 + \frac{2252}{1155}\beta^2 + \frac{90239}{13860}\beta + \frac{281}{99} - \frac{353}{660}\frac{1}{\beta} - \frac{218}{165}\frac{1}{\beta^2} - \frac{1}{2\beta^3}\right)\Big/ 2\beta(1+\beta)^2, \tag{A 22}$$

$$b_2 = \left(-\frac{2203}{396}\beta^5 - \frac{7709}{1386}\beta^4 + \frac{29881}{2772}\beta^3 + \frac{7241}{495}\beta^2 - \frac{3599}{4620}\beta - \frac{4333}{990} + \frac{71}{660}\frac{1}{\beta} - \frac{53}{165}\frac{1}{\beta^2} - \frac{1}{2\beta^3}\right)\Big/ 2\beta(1+\beta)^2, \tag{A 23}$$

$$b_3 = \frac{233}{99}\beta - \frac{32}{99} - \frac{1}{\beta}, \tag{A 24}$$

$$b_4 = -\frac{32}{99}, \tag{A 25}$$

$$b_5 = \frac{8}{35}\left(15\beta^4 - \frac{108}{7}\beta^2 + \frac{73}{21}\right), \tag{A 26}$$

$$b_6 = \frac{2}{147}\left(27\beta^2 - \frac{113}{11}\right), \tag{A 27}$$

$$b_7 = \left(3\beta^4 - \frac{3238}{1155}\beta^2 + \frac{701}{1155}\right)\Big/\beta(1+\beta). \tag{A 28}$$

Appendix B

$\mathscr{G}_n(\mu)$

$$\mathscr{G}_0(\mu) = \mathscr{G}_1(\mu) = 0, \tag{B 1}$$

$$c_0 = -4c, \tag{B 2}$$

$$\begin{aligned}\mathscr{G}_n(\mu) = \frac{g_0^{n-6}(\mu)}{1-\mu^2}\Bigg\{&(n-6)(n-5)(n-4)(n-3)\,g_{n-2} + 2(n^2-9n+21)(1-\mu^2)\,g''_{n-2} \\ &+(1-\mu^2)[(1-\mu^2)\,g''_{n-2}]'' + \sum_{s=0}^{s=n-2}(s-4)\,g'_{n-s-1}[s(s-1)\,g_{s+1} \\ &+(1-\mu^2)\,g''_{s+1}] - \sum_{s=0}^{s=n-2}(n-s-2)\,g_{n-s-1}\left[2s(s-1)\frac{\mu}{1-\mu^2}g_{s+1}\right. \\ &\left.+s(s-1)\,g'_{s+1} + (1-\mu^2)\,g'''_{s+1}\right]\Bigg\} \quad (n \geqslant 2).\end{aligned} \tag{B 3}$$

The pressure p_c calculated from the core-flow stream function $\psi_c(\xi,\mu)$ is

$$-\frac{A^2}{\rho\nu^4}p_c(\xi,\mu) = -\frac{1}{6\xi^6}A_0 - \frac{1}{5\xi^5}A_1 + \sum_{\substack{n=0\\ n\neq 4}}^{n=k}\frac{\xi^{n-4}(A_{n+2}-B_n)}{(n-4)} + K(\mu), \tag{B 4}$$

where

$$A_n(\mu) = \sum_{m=0}^{m=n}\left\{(m-3)\,g'_m\,g'_{n-m} - \frac{(n-m-1)}{1-\mu^2}\,g_{n-m}[(1-\mu^2)\,g''_m + (m-1)\,g_m]\right\}, \tag{B 5}$$

$$B_n(\mu) = -[(1-\mu^2)\,g''_n + (n-2)(n-1)\,g_n], \tag{B 6}$$

and

$$\begin{aligned}K'(\mu) = \frac{1}{1-\mu^2}\Bigg\{&-[(1-\mu^2)g''_4 + 6g_4] \\ &-\sum_{m=0}^{m=6}(m-1)\left[(2m-7)\,g_m\,g'_{6-m} - (5-m)\frac{\mu}{1-\mu^2}g_m\,g_{6-m}\right]\Bigg\}.\end{aligned} \tag{B 7}$$

Unspecified arguments are at a general point μ, and primes denote differentiations with respect to μ. An arbitrary base pressure p_0 is introduced as the constant of integration in the expression for $K(\mu)$. In deriving the above result for p_c, a term $c_6 \ln \xi$ appears, where c_6 is the constant in (5.5) for $n = 6$. This constant must be set equal to zero from arguments presented in § 5.

Appendix C

C.1. $\mathscr{H}_n(\mu)$

$$\omega^4\mathscr{H}_1(\mu) = C_0 - \omega^2 J_0 + 2\beta\tau I_0, \tag{C 1}$$

$$\omega^4\mathscr{H}_2(\mu) = B_2 + C_1 + D_0 - \omega^2(J_1+K_0) + 2\beta\tau(I_1+J_0) + \tau^2 I_0, \tag{C 2}$$

$$\omega^4\mathscr{H}_3(\mu) = B_3+C_2+D_1+E_0-\omega^2(J_2+K_1+M_0) + 2\beta\tau(I_2+J_1+K_0)+\tau^2(I_1+J_0), \tag{C 3}$$

$$\omega^4\mathscr{H}_4(\mu) = B_4+C_3+D_2+E_1+F_0-\omega^2(J_3+K_2+M_1+N_0) + 2\beta\tau(I_3+J_2+K_1+M_0)+\tau^2(I_2+J_1+K_0), \tag{C 4}$$

$$\omega^4\mathscr{H}_5(\mu) = B_5+C_4+D_3+E_2+F_1-\omega^2(J_4+K_3+M_2+N_1) + 2\beta\tau(I_4+J_3+K_2+M_1+N_0)+\tau^2(I_3+J_2+K_1+M_0), \tag{C 5}$$

$$\begin{aligned}\omega^4\mathscr{H}_{n+6}(\mu) = {} & B_{n+6}+C_{n+5}+D_{n+4}+E_{n+3}+F_{n+2} \\ & -\omega^2(J_{n+5}+K_{n+4}+M_{n+3}+N_{n+2}) \\ & +2\beta\tau(I_{n+5}+J_{n+4}+K_{n+3}+M_{n+2}+N_{n+1}) \\ & +\tau^2(I_{n+4}+J_{n+3}+K_{n+2}+M_{n+1}+N_n) \quad (n \geqslant 0),\end{aligned} \tag{C 6}$$

where

$$B_n = \omega^4 \sum_{s=0}^{s=n-2} [(n-s-1)\,h_{n-s-1}h'''_{s+1}-(s-5)\,h'_{n-s-1}h''_{s+1}], \tag{C 7}$$

$$C_n = 4\beta\omega^2\tau \sum_{s=0}^{s=n} [(s-6)\,h'_{n-s}h''_s-(n-s)\,h_{n-s}h'''_s], \tag{C 8}$$

$$\begin{aligned}D_n = \sum_{s=0}^{s=n} \{&(n-s)\,h_{n-s}[\omega^2A'_s-\omega^2\tau^2h''''_s+2\omega^2\tau h''_s+4\beta^2\tau^2h'''_s] \\ &+h'_{n-s}[(s-8)\,\omega^2\tau^2h''_s-4\beta^2\tau^2(s-6)\,h''_s-(s-4)\,\omega^2A_s]\},\end{aligned} \tag{C 9}$$

$$E_n = 2\beta \sum_{s=0}^{s=n} \{(n-s)\,h_{n-s}[\tau^3h''''_s-\tau^2h''_s+A_s-\tau A'_s]+h'_{n-s}[(s-5)\,\tau A_s-(s-7)\,\tau^3h''_s]\}, \tag{C 10}$$

$$F_n = \tau \sum_{s=0}^{s=n} \{(n-s)\,h_{n-s}[2A_s-\tau A'_s]+(s-6)\,\tau h'_{n-s}A_s\}, \tag{C 11}$$

$$I_n = \omega^4 h_n^{\mathrm{iv}}, \tag{C 12}$$

$$J_n = -4\beta\omega^2[\tau h_n^{\mathrm{iv}}+h'''_n], \tag{C 13}$$

$$K_n = (n-4)\,(n-5)\,\omega^2h''_n-2\omega^2(n-5)\,\tau h'''_n+4\beta^2\tau(\tau h''_n)''+\omega^2A''_n, \tag{C 14}$$

$$M_n = 2\beta\tau[2(n-4)\,(\tau h''_n)'-(n-3)\,(n-4)\,h''_n-A''_n], \tag{C 15}$$

$$N_n = (n-3)\,[(n-2)\,A_n-2\tau A'_n], \tag{C 16}$$

$$A_n = (n-1)\,[nh_n-2\tau h'_n]. \tag{C 17}$$

Unspecified arguments are at a general point τ.

C.2. *Construction of particular integrals of* (6.5) *which asymptote to a polynomial of degree* $\leqslant n+1$

From (6.15) and (6.16) $h_0(\tau)$ satisfies (6.11). Assume $h_1, h_2, \ldots, h_{n-1}$ satisfy (6.11). Using these asymptotic values in the equations for B_n, C_n, D_n, E_n, F_n, I_n, J_n, K_n, M_n, N_n given in § 1 of this Appendix we find for large τ:

$$\begin{aligned} B_0 = B_1 = 0, \quad & B_n \sim \tau^{n-1} \quad \text{for} \quad n \geqslant 2, \\ C_0 \sim 0, \quad & C_n \sim \tau^n \quad \text{for} \quad n \geqslant 1, \\ & D_n \sim \tau^{n+1} \quad \text{for} \quad n \geqslant 0, \\ & E_n \sim \tau^{n+3} \quad \text{for} \quad n \geqslant 0, \\ & F_n \sim \tau^{n+3} \quad \text{for} \quad n \geqslant 0, \end{aligned}$$

$$
\begin{array}{lll}
I_0, I_1, I_2 \sim 0, & I_n \sim \tau^{n-3} & \text{for} \quad n \geqslant 3, \\
J_0, J_1 \sim 0, & J_n \sim \tau^{n-2} & \text{for} \quad n \geqslant 2, \\
K_0 \sim 0, & K_n \sim \tau^{n-1} & \text{for} \quad n \geqslant 1, \\
M_0 \sim 0, & M_n \sim \tau^{n} & \text{for} \quad n \geqslant 1, \\
 & N_n \sim \tau^{n+1} & \text{for} \quad n \geqslant 0.
\end{array}
$$

Substituting these expressions in the $\mathscr{H}_n$'s, it is easily verified that, for $n \geqslant 1$,

$$\mathscr{H}_n(\tau, h_0, h_1, \ldots, h_{n-1}) \sim \tau^{n-1}. \tag{C 18}$$

Thus, in general we can write

$$\mathscr{H}_n(\tau, h_0, h_1, \ldots, h_{n-1}) \sim \sum_{s=0}^{s=n-1} d_s \tau^s \quad (n \geqslant 1).\dagger \tag{C 19}$$

To find particular solutions of (6.5) when τ is large, the asymptotic values for the coefficients and $\mathscr{H}_n$ will be used, i.e. we must find solutions of the equation

$$\omega^2 \Pi_n^{\text{iv}} + (n-6)\, a_{01}\, \Pi_n'' = \sum_{s=0}^{s=n-1} d_s \tau^s \quad (n \geqslant 1). \tag{C 20}$$

A particular solution for $n \neq 6$ is

$$\Pi_n(\tau) = \sum_{s=0}^{s=n+1} e_s \tau^s, \tag{C 21}$$

where the constants e_s are related to the constants d_s by the recurrence formulas

$$e_{n+1} = d_{n-1}/(n+1)\, n(n-6)\, a_{01} \quad \text{for} \quad n \geqslant 1, \tag{C 22}$$

$$e_n = d_{n-2}/n(n-1)\,(n-6)\, a_{01} \quad \text{for} \quad n > 1, \tag{C 23}$$

and

$$e_{s+2} = \frac{d_s}{(s+2)\,(s+1)\,(n-6)\, a_{01}} - \frac{\omega^2 (s+4)\,(s+3)\, e_{s+4}}{(n-6)\, a_{01}} \quad \text{for} \quad 0 \leqslant s \leqslant n-3.\ddagger \tag{C 24}$$

e_1 and e_0 are arbitrary constants because a linear function can be introduced as a complementary solution of (C 20). Thus, for $n \neq 6$, $\Pi_n(\tau) \sim \tau^{n+1}$. When $n = 6$, (C 20) can be integrated directly to give

$$\Pi_6(\tau) = \frac{1}{\omega^2} \sum_{s=0}^{s=5} \frac{d_s}{(s+4)\,(s+3)\,(s+2)\,(s+1)} \tau^{s+4} + K_3 \tau^3 + K_2 \tau^2 + K_1 \tau + K_0, \tag{C 25}$$

where the K_n's are arbitrary constants. For Π_6 to asymptote to a polynomial of degree 7 it is necessary that

$$d_5 = d_4 = 0, \tag{C 26}$$

in the asymptotic expansion of $\mathscr{H}_6(\tau)$. A straightforward but tedious calculation shows that (C 26) is satisfied. In the calculation it is necessary to use values for a_{0m} and a_{1m} ($m > 2$), which can be found in terms of a_{01} and a_{11} from the differential equations for $g_0(\mu)$ and $g_1(\mu)$. Therefore, particular integrals of (6.5) can be found which asymptote to a polynomial of degree $\leqslant n+1$ for all n.

† In some cases $d_{n-1} = 0$; e.g. when $n = 1$, $\mathscr{H}_1 \sim 0$.
‡ When $n = 1$, (C 22) determines e_2.